PHYSICS 726
MAGNETOHYDRODYNAMICS

HOMEWORK ASSIGNMENT 2

Due Sept. 27, 2007

1. Prove that, if there is no heating or heat flux, then \( \frac{d}{dt}(p / \rho^\gamma) = 0 \), i.e., the adiabatic law \( p / \rho^\gamma = \text{constant} \) holds for each fluid element.

2. For the case \( \eta = 0 \) (no resistivity), derive an expression for \( \frac{d}{dt}(B / \rho) \). Under what circumstances is \( B / \rho \) constant for a fluid element?

3. Consider the one dimensional domain \( 0 \leq x \leq 1 \). Let the electric charge density be given by \( \rho_q = \Delta n e \left[ \delta(x - x_1) - \delta(x - x_2) \right] \), as shown below.

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0 x1 x2 1
\delta(x-x_1)
\delta(x-x_2)
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a) What is the net electric charge in the domain?

b) Solve for the electrostatic potential \( \phi(x) \), and the electric field \( E = -\nabla \phi \), subject to the boundary conditions \( \phi(0) = \phi(1) = 0 \). Sketch the solutions for both \( \phi \) and \( E \).

c) Comment on the size of the electric field relative to both the net charge in the domain and the concentration of charge density.

4. In HW01 we showed that \( \nabla \cdot (\nabla \alpha \times \nabla \beta) = 0 \), where \( \alpha = \alpha(x,t) \) and \( \beta = \beta(x,t) \). Therefore, the magnetic field can be represented as \( B = \nabla \alpha \times \nabla \beta \). Compute \( \frac{\partial B}{\partial t} \) from Faraday’s law assuming \( E + \nabla \times B = 0 \) (ideal MHD). Show that the resulting relation is satisfied when the field lines move with the fluid, i.e.,

\[
\frac{d\alpha}{dt} \frac{d\beta}{dt} = 0 .
\]
5. Given the viscous stress tensor \( \mathbf{\Pi} = \nu \mathbf{W} \), where \( \mathbf{W} = \nabla \mathbf{V} + \nabla \mathbf{V}^T - (2/3) \nabla \cdot \mathbf{V} \mathbf{I} \), show that \( \text{Trace}(\mathbf{\Pi}) \equiv \Pi_{ii} = 0 \), so that \( \mathbf{\Pi} \) is the traceless part of the total stress tensor. Does this property hold if \( \mathbf{\Pi} = \mathbf{E} : \mathbf{W} \), where \( \mathbf{E} \) is a fourth rank tensor? Why?

6. Write the gyro-viscous stress

\[
\mathbf{\Pi}_G = \frac{\rho}{4\Omega} \left[ \left( \mathbf{\hat{b}} \times \mathbf{W} \right) \cdot \left( \mathbf{1} + 3 \mathbf{\hat{b}} \mathbf{\hat{b}} \right) - \left( \mathbf{1} + 3 \mathbf{\hat{b}} \mathbf{\hat{b}} \right) \cdot \left( \mathbf{W} \times \mathbf{\hat{b}} \right) \right],
\]

in the form \( \mathbf{\Pi}_G = \mathbf{E} : \mathbf{W} \), or \( \Pi_{G,ij} = E_{ijkl} W_{kl} \). Give expressions for the components \( E_{ijkl} \). Show explicitly that

\[
\int_{V} \mathbf{\Pi}_G : \nabla \mathbf{V} dV = 0,
\]

so that the gyro-viscous stress does not dissipate energy.