1. Prove that the ideal MHD force operator is self-adjoint. What are the implications of this for ideal MHD?


Consider the interchange of 2 flux tubes, as shown in the figure. Flux is conserved during the interchange. Before the interchange, element 1 is at \( r_1 \), has volume \( V_1 = A_1L_1 \), pressure \( p_1 \), and flux \( \phi_1 = B_1A_1 \). Element 2 is at \( r_2 \), has volume \( V_2 = A_2L_2 \), pressure \( p_2 \), and flux \( \phi_2 = B_2A_2 \). After the interchange, element 1 is at \( r_2 \), has volume \( V_2 \), pressure \( p_1 \), and flux \( \phi_1 \), while element 2 is at \( r_1 \), has volume \( V_1 \), pressure \( p_2 \), and flux \( \phi_2 \).

A. Show that the change in magnetic energy as a result of the interchange is

\[
\Delta W_M = \frac{1}{2\mu_0} \left( \phi_2^2 - \phi_1^2 \right) \left( \int \frac{dl}{A} - \int \frac{dl}{V} \right).
\]

B. Assume that the interchange is adiabatic, i.e., \( pV^\Gamma = \text{constant} \). Show that the change in internal energy as a result of the interchange is

\[
\Delta W_p = \frac{1}{\Gamma - 1} \left\{ p_2V_2 \left[ \left( \frac{V_2}{V_1} \right)^{\Gamma - 1} - 1 \right] + p_1V \left[ \left( \frac{V_1}{V_2} \right)^{\Gamma - 1} - 1 \right] \right\}.
\]
C. Consider the special case of the interchange of vacuum and fluid elements as sketched (roughly!) in the figure. The equilibrium surface is $S$. The radius of curvature vector for $S$ is $R$; the axis of curvature is $C$. There is no magnetic field in the fluid; the magnetic field in the vacuum is $B$.

The length of tube 1 is $l_1$ and its volume is $V_1$; the length of tube 2 is $l_2$ and its volume is $V_2$. Let $V_1 = V_2$. Use the results of A and B, above, to show that the interchange is stable if $l_2 < l_1$, i.e., if $C$ lies outside the fluid.

D. Now back to the general case. Assume that $\Delta W_M = 0$ (i.e., $\phi_1 = \phi_2 = \phi$). Let $p_1 = p$, $p_2 = p + \delta p$, $V_1 = V$, and $V_2 = V + \delta V$, where $|\delta p / p| << 1$ and $|\delta V / V| << 1$. Show that $\Delta W_p = \delta p \delta V + \Gamma p (\delta V)^2 / V$. Let $p$ decrease outward from the center of the fluid, and assume that $\delta p < 0$ (i.e., tube 1 is further “inside” the fluid than tube 2). Show that a sufficient condition for stability is then $\delta V < 0$. Define the volume of a flux tube as $V = \int A dl$, where $A$ is the cross sectional area. Show that $V = \phi \int dl / B$, and that therefore a sufficient condition for stability is $\delta \int dl / B < 0$, where the integral is taken along the length of the flux tube.

E. Consider the case of a “magnetic bottle” with open field lines, as shown in the figure.

The current vanishes. The flux tubes have lengths $l_1$ and $l_2$, and have equal magnetic flux $\phi$. Show that a sufficient condition for stability is
\[ \int \frac{dl}{rRB^2} > 0 , \]

where the integral is taken along the length of the flux tube. Give a physical interpretation of this condition.

F. Now consider the case of closed flux tubes. The volume of the flux tube is now \( V = \phi \oint dl / B \), where the integral is taken over the closed flux tube. If the flux tube were unconstrained it would tend to expand, since it is filled with pressure. However, since the flux is constant, this means that \( W = -\oint dl / B \) must increase. If we define the “potential” \( W = -\oint dl / B \), then \( V = -\phi W \) so that expansion is characterized by transitions to states of lower \( W \). Assuming that the fluid is adiabatic, show that the change in pressure inside the flux tube is given by

\[ \delta p_I = -\Gamma p \frac{\delta W}{W} . \]

The change in pressure outside the flux tube is given by \( \delta p_E = (dp / dW)\delta W \). Argue that, if the pressure decreases from the “inside” of the fluid to the “outside”, then \( dp / dW > 0 \). Then show that a sufficient condition for stability is

\[ \frac{dp}{dW} < -\frac{\Gamma p}{W} . \]

G. Consider a fluid in the magnetic field of an infinite straight wire carrying current 1. Show that the condition for stability is

\[ -\frac{d\ln p}{d\ln r} < 2\Gamma . \]

H. Consider a fluid in a dipole magnetic field. Then \( B \sim 1/r^3 \) and \( l \sim r \). Show that the condition for stability is

\[ -\frac{d\ln p}{d\ln r} < 4\Gamma . \]

In all cases it is possible to have a stable pressure profile that decreases away from the “inside” of the configuration, as long as the (negative) gradient is not too large. There is at least hope for stable confinement!