1. The eigenvalue problem for the clamped vibrating string is
\[ \frac{d^2 y}{dx^2} + \lambda y = 0 , \]
with \( y(0) = y(1) = 0 \). The exact solution for the fundamental is \( y = \sin \pi x \) and \( \lambda_0 = \pi^2 = 9.8696... \)

A. Transform this into a variational problem by multiplying the differential equation by \( y \), integrating from 0 to 1, and then integrating by parts. The result should be
\[ \lambda = \frac{\int_0^1 (y')^2 \, dx}{\int_0^1 y^2 \, dx} . \]

B. Use the trial function \( y(x) = x(1-x) \) to estimate \( \lambda \). Compare with the exact result.

C. Now use the trial function \( y(x) = x(1-x) + ax^2(1-x)^2 \) to estimate \( \lambda(a) \). Minimize this result with respect to \( a \) (i.e., set \( d\lambda / da = 0 \) and solve for \( a \), then plug this answer back into \( \lambda \)). Compare with the result of B) and with the exact result.

The reference for the following problems is B. B. Kadomtsev, “Hydromagnetic Stability of a Plasma”, in Reviews of Plasma Physics, Vol. 2, p. 153, Consultants Bureau, New York (1966). I apologize that these equations are in cgs units. Results are accepted in any consistent set of units. It looks difficult, but just follow the steps. This is an important MHD calculation (especially #3), and plasma physicist should have gone through it once in their lifetime.

2. Consider the cylindrical Z-pinch with no axial field (\( B = B_0(r)\hat{e}_\phi \)) with the plasma filling the container to the perfectly conducting walls, and consider only axi-symmetric perturbations (independent of \( \theta \)). The equilibrium is a function of \( r \) only.

A. Do we need to allow for \( \nabla \cdot \xi \neq 0 \) in the analysis of the energy principle? Why?

B. Show that \( \delta W \) can be written as a quadratic form
\[ \delta W = \int \left[ a_{11} (\nabla \cdot \xi)^2 + a_{12} (\nabla \cdot \xi) \xi_r + a_{21} \xi_r (\nabla \cdot \xi) + a_{22} \xi_r^2 \right] dV . \]

Give expressions for the \( a_{ij} \).

C. By requiring the principal determinants to vanish, derive the stability condition for axisymmetric (\( m = 0 \)) perturbations:
\[
-\frac{d \ln p}{d \ln r} < \frac{4 \Gamma}{2 + \Gamma \beta} ,
\]
where \( \beta = 8 \pi p / B^2 \) (cgs) is the plasma beta.

D. Now consider non-axisymmetric perturbations (\( m > 0 \)). Let
\[
\xi_r = \hat{\xi}_r(r,z) \sin m \theta, \quad \xi_0 = \hat{\xi}_0(r,z) \cos m \theta, \quad \xi_z = \hat{\xi}_z(r,z) \sin m \theta .
\]

Repeat part B), above. Show that the stability condition is now
\[
-\frac{d \ln p}{d \ln r} < \frac{m^2}{\beta} .
\]
Compare the results of C) and D) as \( \beta \to 0 \) and \( \beta \to \infty \). Which modes are limiting in each case?

3. Consider a surface layer pinch with longitudinal field. In this case there are separate plasma and vacuum regions. The entire axial current flows in the plasma/vacuum boundary. The plasma has radius \( a \) and the wall has radius \( b > a \). Inside the fluid \( p = \text{constant} \) and \( \mathbf{B} = B_{in} \hat{e}_\perp \), with \( B_{in} = \text{constant} \). Outside the fluid, in the vacuum, \( p = 0 \), and \( \hat{\mathbf{B}} = B_o \hat{e}_\phi + B_{in} \hat{e}_z \), with \( B_o = \text{constant} \neq B_{in} \). Assume perturbations of the form \( \exp[i(m \theta + k z)] \).

A. What is the minimizing condition for \( \nabla \cdot \hat{\xi} \)? Why?

B. Show that the ideal MHD wave equation can be written as
\[
-\rho \omega^2 \hat{\xi} = \frac{1}{4 \pi} \left[ \mathbf{Q} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{Q} \right] - \nabla \hat{p} ,
\]
where \( \hat{p} = p + \mathbf{B} \cdot \mathbf{Q} / 4 \pi \).

C. Use \( \nabla \cdot \hat{\xi} = 0 \) (hint! hint!) to show that in the fluid \( \hat{p} \) satisfies the Laplace equation
\[
\nabla^2 \hat{p} = 0 .
\]
Show that this leads to the modified Bessel equation
\[
 r^2 \frac{d^2 \hat{p}}{dr^2} + r \frac{d \hat{p}}{dr} - (k^2 r^2 + m^2) \hat{p} = 0 .
\]
Show that the solution that satisfies \( \hat{p} = \hat{p}(a) \) at \( r = a \), and is finite at \( r = 0 \), is
\[ \tilde{p}(r) = \frac{\tilde{p}(a) I'_m(ka)}{I_m(ka)} . \]

D. Show that the radial displacement at the plasma/vacuum interface is
\[ \begin{aligned} -\omega^2 \rho + \frac{k^2 B^2_m}{4\pi} \xi_r(a) &= -k\tilde{p}(a) \frac{I'_m(ka)}{I_m(ka)}. \end{aligned} \] (3D)

E. Show that the solution in the vacuum satisfies \( \nabla^2 \psi = 0 \), where \( \hat{\mathbf{B}} = \nabla \psi \).

F. Show that the vacuum solution that satisfies \( \partial \psi / \partial r = 0 \) (\( B_r = 0 \)) at \( r = b \) is
\[ \psi_m(r) = \frac{C}{K'_m(kb)} \left[ K'_m(kb) I_m(kr) - I'_m(kb) K_m(kr) \right], \]
where \( C \) is a constant.

G. Show that pressure balance at \( r = a \) can be written as
\[ \tilde{p}(a) = \frac{i}{4\pi} \left( \frac{mB_0(a)}{a} + k B_{z\infty} \right) \psi(a) - \frac{B_0^2(a)}{4\pi a} \xi_r(a). \] (3G)

H. Show that the condition that the total electric field in the vacuum vanish in the frame moving with the displacement at \( r = a \) leads to the condition
\[ \left. \frac{\partial \psi}{\partial r} \right|_{r=a} = i \mathbf{k} \cdot \hat{\mathbf{B}} \xi_r(a), \] (3H)
where \( \mathbf{k} = (m / r)\mathbf{e}_\theta + k\mathbf{e}_z \).

I. Equations (3D), (3G), and (3H) are 3 equations in the 3 unknowns \( \tilde{p}(a), \xi_r(a), \) and \( C \). Set their determinant to zero to find the dispersion relation
\[ 4\pi \rho \omega^2 = k^2 B^2_m - \frac{I'_m(ka)}{I_m(ka)} \left[ \frac{F_m(ka)}{F'_m(ka)} (\mathbf{k} \cdot \hat{\mathbf{B}})^2 + \frac{B_0^2(a)}{a} \right], \]
where
\[ F_m(kr) = \frac{K'_m(kb) I_m(kr) - I'_m(kb) K_m(kr)}{K'_m(kb)}, \]
and
\[ F'_m(kr) = \frac{K'_m(kb) I'_m(kr) - I'_m(kb) K'_m(kr)}{K'_m(kb)}. \]

J. Show that when \( b \to \infty \) the dispersion relation is
\[ 4\pi \rho \omega^2 = k^2 B^2_m - \frac{I'_m(ka)}{I_m(ka)} K'_m(ka) - \frac{B_0^2(a)}{a} \frac{I'_m(ka)}{I_m(ka)}. \]

Give a physical interpretation of each of the three terms on the right hand side.
K. Set \( B_{z_{\text{ext}}} = 0 \) (no external axial field). Show that for \( m = 0 \),

\[
\omega^2 = \frac{k^2 B_{\text{int}}^2}{4\pi\rho} \left[ 1 - \left( \frac{B^0(a)}{B_{\text{int}}} \right)^2 \frac{I'_0(ka)}{kaI_0(ka)} \right].
\]

Show that \( \max\left[ I'_0 / (xI_0) \right] = 1/2 \), and then show the condition for stability for the \( m = 0 \) mode with no external field or conducting wall is \( B_{\text{int}}^2 > B^0(\alpha)/2 \).

L. For the \( m = 1 \) mode with \( B_{z_{\text{ext}}} = 0 \), show that

\[
\omega^2 = \frac{k^2 B_{\text{int}}^2}{4\pi\rho} \left[ 1 + \left( \frac{B^0(a)}{B_{\text{int}}} \right)^2 \frac{I'_0(ka)K_0(ka)}{ka I_1(ka)K'_1(ka)} \right].
\]

For long axial wavelength modes (\( ka \to 0 \)), show that

\[
\omega^2 \approx \frac{k^2 B_{\text{int}}^2}{4\pi\rho} \left[ 1 - \left( \frac{B^0(a)}{B_{\text{int}}} \right)^2 \ln \left( \frac{1}{ka} \right) \right],
\]

so that the \( m = 1 \) mode is always unstable in this limit.

M. Now let \( B_{z_{\text{ext}}} \gg B^0(a) \) (strong external field), \( ka \to 0 \), and set \( m > 0 \). Show that the dispersion relation is

\[
4\pi\rho\omega^2 = k^2B_{\text{int}}^2 + (\mathbf{k} \cdot \hat{\mathbf{B}}(\alpha))^2 - \frac{mB^2(\alpha)}{a^2}.
\] (3M)

Find the value of \( ka \) for which \( \omega^2 \) is minimum. Plug this into the above dispersion relation to show that

\[
\omega^2_{\text{min}} = \frac{B^2(\alpha)}{4\pi\rho a^2} \left[ \frac{m^2B_{\text{int}}^2}{B_{\text{int}}^2 + B^2(\alpha)} - m \right].
\]

Can you give a physical interpretation to each of the terms on the right hand side?

N. Consider the case when \( B_{z_{\text{int}}} = B_{z_{\text{ext}}} = B_z \) (the internal and external axial fields are equal, like a tokamak). Show that in this case

\[
\omega^2_{\text{min}} = \frac{B^2(\alpha)}{4\pi\rho a^2} \left( \frac{m}{2} - 1 \right).
\]

Comment on the stability of the \( m = 1, m = 2 \), and \( m > 2 \) modes.

O. Use the dispersion relation (3M) to find the stability condition

\[
|k| > \frac{B^0(\alpha)}{aB_z}.
\]

Show that, for a quantized system with \( \lambda = L/n \) (where \( L \) is the length of the cylinder), this becomes
\[
\frac{2\pi n}{L} > \frac{B_\theta(a)}{aB_z},
\]

or, for \( n = 1 \) (the longest wavelength),
\[
\frac{2\pi aB_z}{L B_\theta(a)} > 1.
\]

This is the Kruskal-Shafranov stability condition.

P. Show that the Kruskal-Shafranov condition places a limit in the amount of current a pinch can carry without being unstable:
\[
I < \frac{\pi a^2 B_z c}{L}.
\]