9. IDEAL MHD AND THE FROZEN FLUX THEOREM

We have just derived the “complete” MHD fluid mode. We now consider briefly a model that is obtained by ignoring \( \mathbf{P} \) and \( \mathbf{q} \), and setting \( \eta = 0 \). (Note that this is a form of closure.) The medium is thus a perfect electrical conductor, and has no viscosity or thermal conductivity. This is a highly idealized situation, not attainable in nature; it is called ideal MHD. However, it turns out that ideal MHD describes to a remarkably good approximation many of the dynamical properties of hot, strongly magnetized plasmas. This is primarily because most hot plasmas are excellent (although not perfect) conductors of electricity. Ideal MHD is thus of considerable interest. Under these circumstances, the equations of the model reduce to

\[
\begin{align*}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho \mathbf{V} , \\
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) &= -\nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} , \\
\frac{\partial p}{\partial t} &= -\mathbf{V} \cdot \nabla p - \Gamma p \mathbf{V} \cdot \mathbf{V} , \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times (\mathbf{V} \times \mathbf{B}) .
\end{align*}
\]  

These are 8 equations in the 8 unknowns \( \rho, p, \mathbf{V}, \) and \( \mathbf{B} \). We remark that, in this case, Ohm’s law is

\[
\mathbf{E} = -\mathbf{V} \times \mathbf{B} .
\]  

We now derive the most important property of ideal MHD. Consider a closed curve \( C \) within the fluid, and let every point on the curve be moving with the local fluid velocity. We say that \( C \) is co-moving with the fluid, in the Lagrangian sense. Let \( S \) be a surface bounded by \( C \). Define

\[
\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}
\]  

as the flux through \( S \). We ask how \( \Psi \) changes as \( C \) moves with the fluid. As we discussed regarding the Lagrangian change in mass density, \( d\Psi \) consists of two parts:

1. \( d\Psi_1 \), due to the changes in \( \mathbf{B} \) with \( C \) (and \( S \)) held fixed, i.e.,

\[
\left( \frac{d\Psi}{dt} \right)_1 = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\oint_C \mathbf{E} \cdot d\mathbf{l} ;
\]  

2. \( d\Psi_2 \), the amount of magnetic flux swept out by \( C \) as it moves with the fluid. This is calculated as follows. As \( S \) moves about, each line element comprising it moves a distance \( \mathbf{V} dt \), and sweeps out a lateral area \( d\mathbf{S} = \mathbf{V} dt \times d\mathbf{l} \). This is shown in the figure below.
The flux through this area is \( d\Psi_1 = \mathbf{B} \cdot d\mathbf{S} = \mathbf{B} \cdot \mathbf{V} \times d\mathbf{l} dt \), so that

\[
\left(\frac{d\Psi}{dt}\right)_1 = \oint_C \mathbf{B} \cdot \mathbf{V} \times d\mathbf{l} = -\oint_C \mathbf{V} \times \mathbf{B} \cdot d\mathbf{l},
\]

(9.8)

where we have used the properties of the triple vector product.

The total rate of change of flux through \( C \) is then

\[
\frac{d\Psi}{dt} = \left(\frac{d\Psi}{dt}\right)_1 + \left(\frac{d\Psi}{dt}\right)_2,
\]

\[
= -\oint_C \mathbf{E} \cdot d\mathbf{l} - \oint_C \mathbf{V} \times \mathbf{B} \cdot d\mathbf{l},
\]

\[
= -\oint_C (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot d\mathbf{l}.
\]

(9.9)

However, in ideal MHD \( \mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \) (see Equation (9.5)), so that \( d\Psi / dt = 0 \). We conclude that in ideal MHD the magnetic flux through any co-moving closed circuit remains constant. This important result is called the frozen flux condition. It means that the field lines are attached to the fluid (and vice versa); the fluid cannot move across the magnetic field. (However, the fluid is free to slide along \( \mathbf{B} \).) A perpendicular velocity will induce and electric field through \( \mathbf{E} = -\mathbf{V} \times \mathbf{B} \). This will cause a change in \( \mathbf{B} \) through Faraday’s law that is sufficient to make the field lines appear to move with the fluid. If there are both electric and magnetic fields, there will be a perpendicular velocity given by

\[
\mathbf{V}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2}.
\]

(9.10)

This is sometimes called the MHD velocity.

Now consider a volume whose lateral sides are everywhere parallel to the magnetic field, as shown in the figure below.
By construction, \( d\Psi_1 = d\Psi_2 = d\Psi \). If we now consider the case where the volume is long and thin, and make the cross-sectional area infinitesimal, then the volume is called a flux tube. Since in ideal MHD the magnetic field is co-moving with the fluid, the flux tubes also move with the fluid, and the flux through every flux tube in the fluid remains constant as the tube moves about in space. This in turn implies that the magnetic field lines cannot change their topology, or connectivity, since doing so would require violation of the integrity of fluid elements.

If at \( t = 0 \) we start at a point \( \mathbf{x}_0(0) = (x_0(0), y_0(0), z_0(0)) \), and integrate the field line equations

\[
\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} = \frac{dl}{B}
\]

for a distance \( L \), we arrive at a point \( \mathbf{x}_1(0) = (x_1(0), y_1(0), z_1(0)) \); the field line can be thought of as a mapping \( \mathbf{x}_0 \Rightarrow \mathbf{x}_1 \). During a time \( \Delta t \) these two points will have moved to \( \mathbf{x}_0(\Delta t) \) and \( \mathbf{x}_1(\Delta t) \) according to \( d\mathbf{x}_i / dt = \mathbf{V}_i \). If we then integrate Equation (9.11) starting from the point \( \mathbf{x}_0(\Delta t) \), in ideal MHD the trajectory of this field line must also pass through \( \mathbf{x}_1(\Delta t) \), since each point on the field line can be thought of as moving with the local fluid velocity. Thus, in ideal MHD, the mapping \( \mathbf{x}_0(t) \Rightarrow \mathbf{x}_1(t) \) is valid at all times. We say that a field line (or flux tube) retains its identity for all times, i.e., its topology cannot change.

We comment on the boundary conditions for ideal MHD. We consider the boundary of the domain as a “wall” with a unit normal vector \( \mathbf{n} \). From electromagnetic theory we know that the tangential component of \( \mathbf{E} \) and the normal component of \( \mathbf{B} \) must be continuous across an interface. Applying these across the boundary, we have

\[
\mathbf{E}_{\text{inc}}\big|_W = \mathbf{n} \times \mathbf{E}_{\text{inc}}\big|_W \quad \text{is continuous,} \tag{9.12}
\]

and

\[
B_{\text{n}}\big|_W = \mathbf{B} \cdot \mathbf{n}\big|_W \quad \text{is continuous.} \tag{9.13}
\]

If these values are known just outside the boundary, they can be applied just inside the boundary as boundary conditions on the ideal MHD equations. By applying Faraday’s law in the two-dimensional plane of the boundary, we find
If \( \nabla \times \mathbf{E}_{\text{tan}} \big|_w = 0 \), then \( B_n = \) constant and is determined by the initial conditions. A common case is \( B_n = 0 \), so the field does not penetrate the wall. If \( \nabla \times \mathbf{E}_{\text{tan}} \big|_w \neq 0 \), then \( B_n \) evolves in time according to Equation (9.14). If the boundary is a \textit{perfectly conducting wall}, it cannot support an electric field, so \( \mathbf{n} \times \mathbf{E} \big|_w = 0 \). From the ideal MHD Ohm’s law \( \mathbf{E} = -\mathbf{V} \times \mathbf{B} \), we have

\[
\mathbf{n} \times \mathbf{E} = \mathbf{B} (\mathbf{n} \cdot \mathbf{V}) - \mathbf{V} (\mathbf{n} \cdot \mathbf{B}) ,
\]

so that \( \mathbf{V} \) must satisfy the condition

\[
\mathbf{n} \cdot \mathbf{V} \big|_w = \mathbf{n} \cdot \frac{\mathbf{E} \times \mathbf{B}}{B^2} \big|_w + \frac{(\mathbf{n} \cdot \mathbf{B})(\mathbf{B} \cdot \mathbf{V})}{B^2} \big|_w .
\]

The first term on the right hand side is just the Poynting flux arising from an externally applied electric field. The second term introduces complications when \( B_n \neq 0 \), for it allows coupling between the tangential and normal components of the velocity. (It is usually ignored, but this can lead to complications because the boundary conditions are not compatible with Ohm’s law.) We have seen that the density and the total energy satisfy conservation equations, so it is necessary to specify their fluxes at the wall. When \( \mathbf{n} \cdot \mathbf{V} \big|_w = 0 \) both of these fluxes vanish. When \( \mathbf{n} \cdot \mathbf{V} \big|_w \neq 0 \) (as, for example, when there is an electric field at the wall), then, again, the situation is more complicated.

We summarize the \textit{ideal MHD boundary conditions} for the commonly seen case of a \textit{perfectly conducting wall with no normal magnetic field}:

\[
\mathbf{B} \cdot \mathbf{n} \big|_w = 0 ,
\]

\[
\mathbf{n} \times \mathbf{E} \big|_w = 0 ,
\]

and

\[
\mathbf{n} \cdot \mathbf{V} \big|_w = 0 .
\]

These are sufficient to determine the solution of the ideal MHD equations.