10. RESISTIVITY

We just proved that the change in magnetic flux passing through a co-moving closed circuit is

$$\frac{d\Psi}{dt} = -\oint_c (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot d\mathbf{l}. \tag{10.1}$$

Since in ideal MHD $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$, we have $d\Psi / dt = 0$, and we say that the flux is "frozen in" the fluid.

However, in the more general MHD case when the fluid is no longer a perfect electrical conductor, $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$ and

$$\frac{d\Psi}{dt} = -\oint_c \eta \mathbf{J} \cdot d\mathbf{l} \neq 0, \tag{10.2}$$

so that the frozen flux condition no longer applies. This is called resistive MHD. In this case the fluid can "move" separately from and field, and the field lines can "slip across" the fluid. We will eventually see that this can be an important effect, even when the resistivity is small.

In resistive MHD, the combination of Faraday’s law and Ohm’s law becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \nabla \times \left( \frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right). \tag{10.3}$$

The first term is just ideal MHD. The second term is a modification introduced when the electrical conductivity $\sigma = 1 / \eta$ is finite (rather than infinite). When $\eta = \text{constant}$, the last term can be written as

$$\nabla \times \left( \frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right) = \frac{\eta}{\mu_0} \nabla \times \nabla \times \mathbf{B},$$

so that Equation (10.3) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}. \tag{10.4}$$

The effect of resistivity is to introduce diffusion of the magnetic field, with a diffusion coefficient $D_\eta = \eta / \mu_0$ (m$^2$/sec). The characteristic time scale for the diffusion of structures with length scale $L$ is

$$\tau_R = L^2 D_\eta = \mu_0 L^2 / \eta. \tag{10.5}$$
This is called the resistive diffusion time.

We will soon see that the characteristic time scale associated with ideal \((\eta = 0)\) MHD processes is the Alfvén time

\[
\tau_A = \frac{L}{V_A} ,
\]

(10.6)

where \(V_A^2 = B^2 / \mu_0 \rho\) is the square of the Alfvén velocity. The ratio of the resistive and ideal MHD time scales is called the Lundquist number

\[
S = \frac{\tau_R}{\tau_A} = \frac{\mu_0 LV_A}{\eta} .
\]

(10.7)

It turns out that for many (but not all) MHD situations, \(S \gg 1\). The Lundquist number plays an important role in describing the dynamics of hot magnetized plasmas. We will return to the Lundquist number and its importance when we discuss magnetic reconnection later in this course.

We now inquire as to the overall effect of electrical resistivity on plasma confinement. We will discuss confinement in more detail when we discuss MHD equilibrium states. For now, we assume that we can attain a state of quasi-force-balance in which the plasma is contained (or confined) within a magnetic field, with more plasma on the “inside” and more field on the “outside”, as sketched below.

By quasi-force-balance, we mean that the time on which the system evolves is much, much less than then time required for wave propagation across the system (all concepts to be defined later), so that inertia \((\rho d\mathbf{V}/dt)\) can be neglected in the equation of motion. This approach is difficult (but possible) to justify theoretically, but it is useful and we will adopt it here without justification.

With this ansatz, the configuration evolves according to the continuity equation

\[
\frac{\partial n}{\partial t} = -\nabla \cdot n \mathbf{V} = -\nabla \cdot n \mathbf{V}_\perp ,
\]

(10.8)
where $n$ is defined by $\rho = Mn$ ($M$ is the ion mass); the equation of motion neglecting inertia,

$$\nabla p = J \times B \quad ;$$

(10.9)
and the steady state Ohm’s law

$$E = -V \times B + \eta J = \nabla \phi \quad ,$$

(10.10)
where $\phi$ is the scalar potential. Here $E$ can be thought of as an applied electric field. (The last equality assures that $\nabla \times E = 0$, as is required for steady state.) Taking $B \times$ Equation (10.10) and using Equation (10.9), we find the perpendicular velocity to be

$$V_{\perp} = \frac{E \times B}{B^2} - \frac{\eta}{B^2} \nabla p \quad .$$

(10.11)
Substituting this into Equation (10.8), we have

$$\frac{\partial n}{\partial t} = \nabla \cdot \left( \frac{\eta n}{B^2} \nabla p \right) - \nabla \cdot \left( n \frac{\nabla \phi \times B}{B^2} \right) \quad .$$

(10.12)
The primary effects are illustrated for the isothermal case $p = nT$, with $T = \text{constant}$. Then the density evolves according to

$$\frac{\partial n}{\partial t} = \nabla \cdot (D_n \nabla n) - \nabla \cdot (n V_E) \quad ,$$

(10.13)
where $D_n = \eta n T / B^2$ is a diffusion coefficient, and $V_E = \nabla \phi \times B / B^2$ is the MHD velocity resulting from the applied electric field. The first term represents diffusion across the confining field lines, and the second is a generally inward convection. The characteristic time for diffusion of the plasma across the size of the system is $\tau_n = B^2 L^2 / \eta n T$; we might expect the plasma to be substantially lost from the system on this time scale in the absence of an applied electric field. Therefore, resistivity has a negative effect on plasma confinement.

The concepts illustrated above form the basis of what are called “1-½ dimensional transport models”. When this calculation is done in a torus with all the geometric complexity that ensues, it is called Pfirsch-Schlüter transport.