13. REDUCED MHD

One often encounters situations in which the magnetic field is strong and *almost* uni-directional. Since a constant field does not produce a current density, these fields are sometimes said to be *almost potential*. Examples are the magnetic fields in loops in the solar corona, and tokamaks. What current density exists arises from small variations of the field from uniformity. This situation is sketched in the figure.

Historically, the development of the reduced MHD model was motivated by some properties of the full MHD equations that we have not yet discussed. These involve issues of force balance and time and space scales associated with various manifestations of plasma dynamics. It is therefore not clear when it is appropriate to introduce reduced MHD into this course of study. I have chosen to present it here, at the end of the development of various MHD models, rather than waiting until the details of the important waves and instabilities have been worked out. This will require us to look ahead a little and anticipate some of the important results. The derivation given here will therefore be more heuristic than formal. I hope this does not lead to too much confusion.

When we say that the field is strong, we mean that both the kinetic and internal energy densities are much smaller than the magnetic energy density, i.e.,

$$\rho V^2 \sim p \ll \frac{B^2}{2\mu_0}.$$  \hfill (13.1)

Since the magnetic field is almost uniform and uni-directional, the field has one almost uniform component ($B_z$, say, taken to be positive) that is much larger than the other components. It is customary to denote these other components collectively by the notation $B_\perp$, meaning the components of $B$ perpendicular to the strong, nearly uniform component. [In this choice of coordinate system, these are the $(x,y)$ components.] The magnetic field is thus characterized by the condition

$$\frac{B_\perp}{B_z} \sim \epsilon \ll 1.$$  \hfill (13.2)
We will seek a simplified MHD model that describes the dynamics under these conditions. Formally, the variables appearing in the MHD equations are ordered as some power of the small parameter $\varepsilon$. This ansatz is introduced into the MHD equations, and only the lowest powers of $\varepsilon$ are retained. The formal procedure also removes the fastest time scale from the problem. The resulting equations have been found to be extremely useful for both analytic and numerical calculations. The model is called reduced MHD. It describes the dynamics of system in the plane perpendicular to the mean field.

The condition given in Equation (13.2) implies the ordering
\[ B_\perp \sim \varepsilon, \quad B_z \sim 1. \]  
(13.3)

To illustrate the formalism, we calculate the unit vector parallel to the magnetic field:
\[
\hat{b} = \frac{B}{B} = \frac{B_\perp + B_z \hat{e}_z}{\sqrt{B_\perp^2 + B_z^2}},
\]
\[
= \varepsilon \frac{B_\perp + B_z \hat{e}_z}{B_z \left(1 + \varepsilon^2 \frac{B_\perp^2}{B_z^2}\right)} \approx \varepsilon \frac{B_\perp + B_z \hat{e}_z \left(1 - \frac{1}{2} \varepsilon^2 \frac{B_\perp^2}{B_z^2}\right)}{B_z},
\]
\[
= \hat{e}_z + \varepsilon \frac{B_\perp}{B_z} + O(\varepsilon^2) \approx \hat{e}_z,
\]
(13.3)
to lowest order in $\varepsilon$.

We now assume that the dynamics of the fluid and the field in the plane perpendicular to the mean field lead to approximate energy equipartition, i.e.,
\[ \rho V_\perp^2 \sim p \sim \frac{B_\perp^2}{2\mu_0}. \]  
(13.4)

Since $B_\perp \sim \varepsilon$, we require for consistency
\[ V_\perp \sim \varepsilon, \quad p \sim \varepsilon^2. \]  
(13.5)

We will see that the most important motions in systems of this sort have little spatial variation along the mean magnetic field. Most of their spatial structure is in the plane perpendicular to the mean field [the $(x,y)$ plane]. Using this hindsight, we introduce the ordering
\[
\frac{\lambda_\parallel}{\lambda_\perp} = \frac{k_\parallel}{k_\perp} \sim \varepsilon,
\]
(13.6)
which implies that
\[ V_\perp \sim 1, \quad \frac{\partial}{\partial z} \sim \varepsilon. \]  
(13.7)

Again with hindsight, we assume that the dynamics parallel to the mean magnetic field occurs on a much shorter time scale than the dynamics in the perpendicular plane. (For example, sound waves will propagate rapidly along the field and smooth out
significant variations in that direction.) In this case, we expect approximate force balance to be maintained in the parallel direction on the time scale of the perpendicular dynamics, i.e.,

\[ \hat{b} \cdot \nabla \left( p + \frac{B^2}{2\mu_0} \right) = 0 \]  

so that \( \frac{dV_z}{dt} = 0 \), or \( V_z \approx \text{constant} \); we choose \( V_z = 0 \). Using Equation (13.3), this implies

\[ \frac{\partial p}{\partial z} + \frac{1}{\mu_0} B_z \frac{\partial B_z}{\partial z} = 0 \]  

(13.9)

Since \( B_z \) is almost uniform, we can write \( B_z = B_{z0} + \tilde{B}_z(x,y,z) \), where \( B_{z0} = \text{constant} \). Then

\[ \frac{\partial p}{\partial z} + \frac{1}{\mu_0} B_{z0} \frac{\partial \tilde{B}_z}{\partial z} = 0 \]  

or \( p \sim (B_{z0} / \mu_0) \tilde{B}_z \), which, in light of (13.5), yields the ordering

\[ \tilde{B}_z \sim \varepsilon^2 \]  

(13.11)

Finally, we are interested in situations in which the resistivity is small, so we order

\[ \eta \sim \varepsilon \]  

(13.12)

The reduced MHD ordering is then summarized as

\[ V_z = 0, \quad \nabla \perp \sim 1, \quad \frac{\partial}{\partial z} \sim \varepsilon, \quad \eta \sim \varepsilon, \]  

\[ V_\perp \sim \varepsilon, \quad p \sim \varepsilon^2, \quad \tilde{B}_z \sim \varepsilon^2 \]  

(13.13)

It is also customary to take \( \rho = \rho_0 = \text{constant} \sim 1 \).

We now proceed with the derivation. The current density is

\[ \mu_0 J = \nabla \times B \]  

\[ = \left( \nabla \perp + \varepsilon \hat{e}_z \frac{\partial}{\partial z} \right) \times \left( \varepsilon B_\perp + B_{z0} \hat{e}_z \right) \]  

\[ = \varepsilon \nabla \perp \times B_\perp + O(\varepsilon^3) \]  

so that

\[ \mu_0 J_z \sim \varepsilon, \quad \mu_0 J_\perp \sim \varepsilon^3 \]  

(13.14)

The magnetic field is written as

\[ B = \hat{e}_z \times \nabla \psi + B_{z0} \hat{e}_z \]  

(13.15)

and the condition \( \nabla \cdot B = 0 \) is satisfied to lowest order in \( \varepsilon \), i.e.,
\[ \nabla \cdot \mathbf{B} = \nabla \cdot \left( \mathbf{B}_\perp + B_{z0} \hat{e}_z \right) , \]
\[ = \nabla \cdot \mathbf{B}_\perp , \]
\[ = \nabla \cdot \left( \hat{e}_z \times \nabla \psi \right) , \]
\[ = \hat{e}_z \cdot \nabla \times \nabla \psi + \nabla \psi \cdot \nabla \times \hat{e}_z , \]
\[ = 0 . \]

In this representation, the current density is
\[ \mu_0 \mathbf{J} = \nabla \times (\hat{e}_z \times \nabla \psi) = \hat{e}_z \nabla^2 \psi + O(\varepsilon) , \]
so that
\[ \mu_0 J_z = \nabla^2 \psi . \quad (13.16) \]

The flux function \( \psi \) is related to the vector potential by
\[ \mathbf{B} = \varepsilon \mathbf{B}_\perp + B_{z0} \hat{e}_z = \nabla \times \mathbf{A} , \]
\[ = \left( \nabla \perp + \varepsilon \hat{e}_z \frac{\partial}{\partial z} \right) \times \left( \mathbf{A}_\perp + A_z \hat{e}_z \right) , \]
\[ = \nabla \times \mathbf{A}_\perp - \hat{e}_z \times \nabla A_z + \varepsilon \frac{\partial}{\partial z} (\hat{e}_z \times \mathbf{A}_\perp) , \]
\[ = \hat{e}_z \times \nabla \left( -A_z \right) + \nabla \times \mathbf{A}_\perp , \]

so that \( \psi = -A_z . \) Further, since \( B_z = B_{z0} + \varepsilon^2 \hat{B}_z \), we have \( \nabla \times \mathbf{A}_\perp = 0 \), and we choose \( \mathbf{A}_\perp = 0 . \)

Now look at the induction equation (Faraday’s law and Ohm’s law). Since \( \partial \mathbf{B} / \partial t = -\nabla \times \mathbf{E} \) and \( \mathbf{E} = -\nabla \times \mathbf{B} + \eta \mathbf{J} \), using the choices of the previous paragraph we have
\[ \hat{e}_z \frac{\partial A_z}{\partial t} = \mathbf{V} \times \mathbf{B} - \eta \mathbf{J} - \nabla \chi , \quad (13.17) \]
where \( \chi \) is the scalar potential. The parallel and perpendicular components of Equation (13.17) are
\[ \frac{\partial \psi}{\partial t} = \hat{e}_z \cdot \mathbf{V} \times \mathbf{B} - \eta J_z - \frac{\partial \chi}{\partial z} , \quad (13.18) \]
and
\[ 0 = (\mathbf{V} \times \mathbf{B})_\perp - \nabla \chi . \quad (13.19) \]

In Equation (13.18) we have used the fact that \( \eta J_\perp \sim \varepsilon^3 \). Now, since \( V_z = 0 \), \( \mathbf{V} \times \mathbf{B} = \mathbf{V}_\perp \times \mathbf{B}_\perp + B_{z0} \mathbf{V}_\perp \times \hat{e}_z \), and therefore
\[ \hat{e}_z \cdot \mathbf{V} \times \mathbf{B} = \hat{e}_z \cdot \mathbf{V}_\perp \times \mathbf{B}_\perp , \quad (13.20) \]
and
\[(\mathbf{V} \times \mathbf{B})_\perp = B_{z0} \mathbf{V}_\perp \times \hat{e}_z \quad . \tag{13.21}\]

Then using Equations (13.19) and (13.21), \( \mathbf{V}_\perp \times \hat{e}_z = \nabla \chi \), and so
\[
\mathbf{V}_\perp = \hat{e}_z \times \nabla \phi \quad , \tag{13.22}\]
where \( \phi = \chi / B_{z0} \) is the *stream function* for the velocity. With this representation we see that
\[
\nabla_\perp \cdot \mathbf{V}_\perp = \nabla_\perp \cdot (\hat{e}_z \times \nabla \phi) = 0 \quad , \tag{13.23}\]
so that the flow is incompressible in the perpendicular plane.

Using Equations (13.15), (13.16) and (13.20), we see that Equation (13.18), the parallel component of the induction equation, becomes
\[
\frac{\partial \psi}{\partial t} = -\mathbf{V}_\perp \cdot \nabla \psi + \frac{\eta}{\mu_0} \nabla^2 \psi - B_{z0} \frac{\partial \phi}{\partial z} \quad . \tag{13.24}\]

Equation (13.24) describes the evolution of the magnetic field. It contains the unknown potential functions \( \psi \) and \( \phi \). It remains to find and equation for the evolution of the flow. For this type of problem it is convenient to use the \( z \)-component of the curl of the equation of motion, i.e.,
\[
\hat{e}_z \cdot \rho_0 \left[ \frac{\partial}{\partial t} \nabla \times \mathbf{V} + \nabla \times (\mathbf{V} \cdot \nabla \mathbf{V}) \right] = \hat{e}_z \cdot \nabla \times (\mathbf{J} \times \mathbf{B}) \quad . \tag{13.25}\]

We use the useful identity \( \mathbf{V} \cdot \nabla \mathbf{V} = \nabla (V^2 / 2) - \mathbf{V} \times \nabla \times \mathbf{V} \) and define the *vorticity* as \( \mathbf{\omega} = \nabla \times \mathbf{V} \). Then
\[
\nabla \times (\mathbf{V} \cdot \nabla \mathbf{V}) = -\nabla \times (\mathbf{V} \times \mathbf{\omega}) \quad ,
\]
\[
= -\mathbf{\omega} \cdot \nabla \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{\omega} + \mathbf{\omega} \nabla \cdot \mathbf{V} \quad .
\]

Since \( V_z = \nabla \cdot \mathbf{V} = 0 \), the \( z \)-component of this expression is \( \mathbf{V} \cdot \nabla \mathbf{\omega} \), where \( \mathbf{\omega} = \hat{e}_z \cdot \nabla \times \mathbf{V} \) is the \( z \)-component of the vorticity, or, in light of Equation (13.22), \( \mathbf{\omega} = \nabla^2_z \phi \). Then Equation (13.25) can be written as
\[
\rho_0 \left[ \frac{\partial \mathbf{\omega}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{\omega} \right] = \hat{e}_z \cdot \nabla \times (\mathbf{J} \times \mathbf{B}) \quad . \tag{13.26}\]

The right hand side simplifies according to
\[
\nabla \times (\mathbf{J} \times \mathbf{B}) = \mathbf{B} \cdot \nabla (J_z \hat{e}_z) \quad ,
\]
since \( \nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{J} = 0 \), and \( J_z \partial B_z / \partial z = 0 \) to lowest order in \( \varepsilon \). The resulting equation is
\[
\rho_0 \left[ \frac{\partial \mathbf{\omega}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{\omega} \right] = \mathbf{B} \cdot \nabla (J_z \hat{e}_z) \quad . \tag{13.27}\]

The reduced MHD model therefore consists of Equations (13.24) and (13.27):
\[
\frac{\partial \psi}{\partial t} = -\mathbf{V}_\perp \cdot \nabla \psi + \frac{\eta}{\mu_0} \nabla^2 \psi - B_{z0} \frac{\partial \phi}{\partial z}, \quad (13.28)
\]

and
\[
\rho_0 \left( \frac{\partial \omega}{\partial t} + \mathbf{V} \cdot \nabla \omega \right) = \mathbf{B} \cdot \nabla (J_z \hat{e}_z). \quad (13.29)
\]

These equations are closed by the subsidiary relations
\[
\omega = \nabla^2 \phi, \quad (13.30)
\mu_0 J_z = \nabla^2 \psi, \quad (13.31)
\]

and
\[
\mathbf{V}_\perp = \hat{e}_z \times \nabla \phi. \quad (13.32)
\]

We note the following:

1. Reduced MHD consists of 6 equations in the 6 unknowns \(\psi, \omega, \phi, J_z, \) and \(\mathbf{V}_\perp.\)

2. We need only deal with scalar functions.

3. There are no parallel dynamics; these fast time scales have been ordered out of the problem.

4. Two of the equations [(13.30) and (13.31)] are of the Poisson type.

5. The pressure does not enter the equations; it has been ignored since \(\beta = 2\mu_0 p / B_z^2 \sim \epsilon^2.\) However, the model can be extended to include the ordering \(\beta \sim \epsilon\) (called the “finite-\(\beta\)” equations).

6. Reduced MHD forms the basis of much of modern tokamak theory.

That’s all we’ll say about reduced MHD.