15. SIMPLE MHD EQUILIBRIA

In this Section we will examine some simple examples of MHD equilibrium configurations. These will all be in cylindrical geometry. They form the basis for more complicated equilibrium states in toroidal geometry.

Many MHD equilibrium configurations (including tokamaks, spheromaks, and RFPs) are based on the pinch effect, which results from the attractive nature of parallel currents (exceptions are mirrors and stellarators). Consider two elements carrying currents $J_1$ and $J_2$ in the $z$-direction, as shown in the figure.

The magnetic field $B_1$ produced by $J_1$ encircles element 1 according to the right hand rule, and similarly for $B_2$ and $J_2$. The Lorentz force $J_1 \times B_2$ acting in element 1 is directed toward element 2. Similarly, the Lorentz force $J_2 \times B_1$ acting on element 2 is directed toward element 1.

If the current $J$ is distributed continuously in space, the net effect of the Lorentz force will be to pull the fluid together, compressing it and thereby increasing the pressure. This process will cease when the increase in the pressure force tending to expand the fluid just balances the Lorentz force tending to compress the fluid, or $\nabla p = J \times B$. If the current flows in a column, the column will tend to contract, or pinch, in a direction perpendicular to its axis until the equilibrium condition is reached. This is called the pinch effect, and is shown in the figure.
Equilibrium configurations based on the pinch effect are named after the direction of the current, not the magnetic field. We will now examine several of these in cylindrical geometry.

In the *Theta Pinch* (or *θ*-Pinch), the current flows only in the azimuthal, or *θ*, direction. This produces a magnetic field in the *z*-direction, as shown in the figure.

The magnetic field is a combination of a uniform, externally generated field and the field produced by the current. We take *B*<sub>z</sub> to be in the positive *z* -direction, and *J*<sub>θ</sub> to be in the negative *θ*-direction. We assume that all quantities are functions of *r* only. The Lorentz force is then radially inward, i.e.,

\[
\mathbf{J} \times \mathbf{B} = -J_\theta \hat{\mathbf{e}}_\theta \times B_z \hat{\mathbf{e}}_z = -J_\theta B_z \hat{\mathbf{e}}_r .
\]  

(15.1)

We will generally use three equations to analyze an equilibrium configuration. These are:  
1. \( \nabla \cdot \mathbf{B} = 0 \);  
2. Ampère’s law, \( \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \); and,  
3. Force balance, \( \nabla p = \mathbf{J} \times \mathbf{B} \).

These are now considered for the case of the Theta-Pinch.

1. \( \nabla \cdot \mathbf{B} = 0 \). In cylindrical geometry, this is

\[
\frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0 .
\]  

(15.2)

Since the configuration depends only on *r*, and \( B_r = 0 \), we require

\[
\frac{\partial B_z}{\partial z} = 0 ,
\]  

(15.3)

which is satisfied automatically.

2. *Ampère’s law*, \( \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \). Under these conditions, this becomes

\[
\mu_0 J_\theta = -\frac{dB_z}{dr} .
\]  

(15.4)

3. *Force balance*, \( \nabla p = \mathbf{J} \times \mathbf{B} \). This becomes
\[
\frac{dp}{dr} = J_\theta B_z .
\]  
(15.5)

Substituting Equation (15.4) into Equation (15.5), we have
\[
\frac{dp}{dr} = B_z \left( -\frac{1}{\mu_0} \frac{dB_z}{dr} \right) = -\frac{d}{dr} \left( \frac{B_z^2}{2\mu_0} \right),
\]
or
\[
\frac{d}{dr} \left( p + \frac{B_z^2}{2\mu_0} \right) = 0 .
\]  
(15.6)

The second term in parentheses is the magnetic pressure. This can be integrated to yield
\[
p + \frac{B_z^2}{2\mu_0} = \frac{B_0^2}{2\mu_0},
\]  
(15.7)
so that \( B_z = B_0 \) when \( p = 0 \), i.e., outside the fluid. The constant \( B_0 \) is thus the externally generated component of the axial magnetic field. Note that Equation (15.7) is a single equation containing two unknowns, \( B_z \) and \( p \). We are free to specify one and then determine the other. This will be a general property of MHD equilibria. An example of a solution of Equation (15.7) is
\[
p(r) = p_0 e^{-r^2/a^2},
\]  
(15.8)
and
\[
B_z(r) = B_0 \left( 1 - \beta_0 e^{-r^2/a^2} \right)^{1/2},
\]  
(15.9)
where \( \beta_0 = 2\mu_0 p_0 / B_0^2 \) and \( r = a \) is the radius of the outer boundary. These solutions are sketched (very roughly!) in the figure.

We now consider the linear Z-pinch. The current now flows in the \( z \)-direction, and the magnetic field is in the \( \theta \)-direction, as shown in the figure.
We again assume that there is only \( r \), and proceed as with the \( \theta \)-pinch.

1. \( \nabla \cdot \mathbf{B} = 0 \).

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r B_r \right) + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0 \quad . (15.10)
\]

We have \( B_r = B_z = 0 \), so we require

\[
\frac{\partial B_\theta}{\partial \theta} = 0 \quad , (15.11)
\]

which is automatically satisfied if \( B_\theta = B_\theta(r) \).

2. Ampère’s law, \( \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \).

\[
\mu_0 J_z = \frac{1}{r} \frac{d}{dr} \left( r B_\theta \right) \quad . (15.12)
\]

3. Force balance, \( \nabla p = \mathbf{J} \times \mathbf{B} \).

\[
\frac{dp}{dr} = -J_z B_\theta \quad . (15.13)
\]

Using Equation (15.12),

\[
\frac{dp}{dr} = - \frac{B_\theta}{\mu_0 r} \frac{d}{dr} \left( r B_\theta \right) \quad ,
\]

\[= - \frac{B_\theta}{\mu_0} \frac{dB_\theta}{dr} - \frac{B_\theta^2}{\mu_0 r} \quad ,
\]

or

\[
\frac{d}{dr} \left( p + \frac{B_\theta^2}{2 \mu_0} \right) = - \frac{B_\theta^2}{\mu_0 r} \quad . (15.14)
\]

This looks like the result for the \( \theta \)-pinch, Equation (15.16), with the addition of a term on the right hand side. This term is called the hoop stress, and arises from the curvature of the magnetic field lines. (In the \( \theta \)-pinch, the field lines are straight.)
Consider the curve shown in the figure.

Let \( s \) be the distance along the curve, and define \( \mathbf{t} \) as a unit vector tangent to the curve. The \textit{curvature vector} is then defined as

\[
\mathbf{\kappa} = \frac{d\mathbf{t}}{ds},
\]

the rate of change of the tangent vector as we move along the curve. The \textit{radius of curvature} at a point \( s \) is defined as

\[
R_c = \frac{1}{|\mathbf{\kappa}|}.
\]

If the curve is a magnetic field line, the unit tangent vector is \( \hat{\mathbf{b}} = \mathbf{B} / B \), and \( d / ds = \hat{\mathbf{b}} \cdot \nabla \), so the \textit{curvature of a magnetic field line} is

\[
\mathbf{\kappa} = \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}.
\]

A straightforward calculation yields

\[
\hat{\mathbf{b}} \times (\nabla \times \hat{\mathbf{b}}) = -\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} = -\mathbf{\kappa}.
\]

Then another straightforward calculation using Ampère’s law and force balance leads to

\[
\mathbf{\kappa} = \frac{\mu_0}{B^2} \nabla p + \frac{1}{B} \nabla \cdot B
\]

\[
= \frac{\mu_0}{B^2} \nabla \left( p + \frac{B^2}{2\mu_0} \right),
\]

or
\[ \nabla \left( p + \frac{B^2}{2\mu_0} \right) = \frac{B^2}{\mu_0} \kappa . \] (15.17)

If \( \kappa = 0 \), as in the \( \theta \)-pinch, we obtain Equation (15.6). For the case of the \( z \)-pinch, we have

\[
\kappa = \hat{b} \cdot \nabla \hat{b} = \frac{B_\theta \hat{e}_\theta}{B_\theta} \nabla \left( \frac{B_\theta \hat{e}_\theta}{B_\theta} \right) ,
= \hat{e}_\theta \cdot \nabla \hat{e}_\theta = \hat{e}_\theta \cdot \left( \frac{\hat{e}_\theta}{r} \frac{\partial \hat{e}_\theta}{\partial r} \right) ,
= \frac{1}{r} \frac{\partial \hat{e}_\theta}{\partial r} = -\frac{\hat{e}_r}{r} . \] (15.18)

Then Equation (15.17) becomes

\[ \nabla \left( p + \frac{B^2}{2\mu_0} \right) = -\frac{B^2}{\mu_0 r} \hat{e}_r , \]

or, for our one-dimensional configuration,

\[ \frac{d}{dr} \left( p + \frac{B^2}{2\mu_0} \right) = -\frac{B^2}{\mu_0 r} , \] (15.19)

which agrees with Equation (15.14). The hoop stress, or tension force, balances the gradient of the total pressure. Again, this is one equation in two unknowns. One can be specified arbitrarily.

The case that contains both \( B_\theta (r) \) and \( B_z (r) \) (and, consequently, both \( J_\theta \) and \( J_z \)) is called the general screw pinch, because the field lines wrap around the cylinder in a helical fashion, like the threads on a screw. For this configuration:

1. \( \nabla \cdot \mathbf{B} = 0 \).
2. Ampère’s law, \( \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \).
   \[ \mu_0 J_\theta = -\frac{dB_\theta}{dr} , \] (15.21)
   and
   \[ \mu_0 J_z = \frac{1}{r} \frac{d}{dr} (rB_\theta) . \] (15.22)
3. Force balance, \( \nabla p = \mathbf{J} \times \mathbf{B} \).
\[
\frac{dp}{dr} = J_\theta B_z - J_z B_\theta ,
\]
\[
= -\frac{d}{dr} \left( \frac{B_z^2}{2\mu_0} \right) - \frac{d}{dr} \left( \frac{B_\theta^2}{2\mu_0} \right) - \frac{B_\theta^2}{\mu_0 r} ,
\]
or
\[
\frac{d}{dr} \left( p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) = -\frac{B_\theta^2}{\mu_0 r} .
\] (15.23)

We now have one equation in three unknowns, so that two of functions can be specified.

We will follow the same procedure for analyzing the more complicated situation of toroidal equilibrium.

Finally, we remark that in each of the examples considered in this Section, the cylinder is infinitely long in the z-direction, i.e., each example is purely two-dimensional. Recall that the Virial Theorem proven in Section 14 assumed that a surface of integration could be taken completely outside the fluid. This is clearly impossible if the fluid extends to infinity in some direction. We thus do not expect the Virial Theorem to apply to these simple examples.