The ideal MHD equation of motion is
\[ \rho_0 \ddot{\xi} = F\{\xi\} \quad . \quad (22.1) \]
We will now prove that the ideal MHD force operator \( F\{\xi\} \) is self-adjoint. That is, we will demonstrate that for any two vector fields \( \xi \) and \( \eta \) satisfying the same boundary conditions,
\[ \int dV \eta \cdot F\{\xi\} = \int dV \xi \cdot F\{\eta\} \quad . \quad (22.2) \]

**Proof:** The total energy of the system is
\[ U = \int \left( \frac{1}{2} \rho V^2 + \frac{B^2}{2\mu_0} + \frac{p}{\Gamma-1} \right) dV \quad . \quad (22.3) \]
We have seen that this quantity is conserved in ideal MHD, so that
\[ U = K + \delta W = \text{constant} \quad . \quad (22.4) \]
Here, \( K \) is the kinetic energy, and \( \delta W \) is the change in the potential energy of the system, as a result of the displacement. Therefore
\[ \dot{U} = \dot{K} + \delta \dot{W} = 0 \quad . \quad (22.5) \]
Now the kinetic energy is
\[ K = \frac{1}{2} \int \rho_0 V^2 dV = \frac{1}{2} \int \rho_0 \dot{\xi}^2 dV \quad , \quad (22.6) \]
or
\[ K\{\xi,\dot{\xi}\} = \frac{1}{2} \int \rho_0 \dot{\xi} \cdot \ddot{\xi} dV \quad , \quad (22.7) \]
so that \( K \) is a symmetric functional of \( \dot{\xi} \). Then the rate of change of kinetic energy is
\[ \dot{K} = \int \rho_0 \dot{\xi} \cdot \ddot{\xi} dV = 2K\{\xi,\dot{\xi}\} \quad , \quad (22.8) \]
because of the symmetry of the arguments. In light of Equation (22.1), this can be written as
\[ \dot{K} = 2K\left\{ \frac{1}{\rho_0} F\{\xi\},\dot{\xi} \right\} \quad , \quad (22.9) \]
or, in light of Equation (22.6),
\[ \dot{K} = -\delta \dot{W} \quad , \quad (22.10) \]
Let \( r_0 \) stand for the equilibrium position of the system and let \( W(r_0) \) be the equilibrium potential energy of the system. The potential energy after a small displacement \( \xi \) will change according to

\[
W(r_0 + \xi) = W(r_0) + \sum_i \left( \frac{\partial W}{\partial r_i} \right)_0 \xi_i + \frac{1}{2} \sum_i \sum_j \left( \frac{\partial^2 W}{\partial r_i \partial r_j} \right)_0 \xi_i \xi_j + ... \:
\tag{22.11}
\]

But equilibrium is an extremum of the potential energy, so that \( \frac{\partial W}{\partial r_i} \big|_{r_0} = 0 \). Therefore

\[
\delta W = W(r_0 + \xi) - W(r_0)
\]

\[
= \frac{1}{2} \sum_i \sum_j \left( \frac{\partial^2 W}{\partial r_i \partial r_j} \right)_0 \xi_i \xi_j
\]

\[
= \delta W \{ \xi, \xi \}
\tag{22.12}
\]

which is a symmetric quadratic form, so that \( \delta W \) is also a symmetric function of its arguments. For any two vector functions \( \xi \) and \( \eta \) satisfying the same boundary conditions,

\[
\delta W \{ \xi, \eta \} = \delta W \{ \eta, \xi \}
\tag{22.13}
\]

Applying this to Equation (22.10), we have

\[
\dot{K} = -2\delta W \{ \xi, \xi \}
\tag{22.14}
\]

or, using Equation (22.9),

\[
K \left\{ \frac{1}{\rho_0} \mathbf{F} \{ \xi \}, \dot{\xi} \right\} = -\delta W \{ \xi, \xi \}
\tag{22.15}
\]

Now, it is important to recognize that \( \xi \) and \( \dot{\xi} \) are independent functions. To avoid confusion on this point, we replace \( \dot{\xi} \) by \( \eta \), so that Equation (22.15) becomes

\[
K \left\{ \frac{1}{\rho_0} \mathbf{F} \{ \xi \}, \eta \right\} = -\delta W \{ \xi, \xi \}
\tag{22.16}
\]

Equation (22.16) must hold for arbitrary \( \xi \) and \( \eta \). In particular, it must hold if we interchange \( \xi \) and \( \eta \), i.e.,

\[
K \left\{ \frac{1}{\rho_0} \mathbf{F} \{ \eta \}, \xi \right\} = -\delta W \{ \eta, \xi \}
\tag{22.17}
\]

Using Equations (22.13) and (22.16),

\[
\delta W \{ \xi, \eta \} = \delta W \{ \eta, \xi \}
\]
\[-K \left\{ \frac{1}{\rho_0} F\{\xi\}, \eta \right\} \cdot \]

Therefore, from Equation (22.9),

\[K \left\{ \frac{1}{\rho_0} F\{\eta\}, \xi \right\} = -\delta W \{\xi, \eta\} ,\]

\[= K \left\{ \frac{1}{\rho_0} F\{\xi\}, \eta \right\} .\]  \hspace{1cm} (22.19)

But, by definition,

\[K \{\xi, \eta\} = \frac{1}{2} \int dV \rho_0 \xi \cdot \eta ,\]

so that

\[\int dV \eta \cdot F\{\xi\} = \int dV \xi \cdot F\{\eta\} ,\]  \hspace{1cm} (22.20)

which is identical to Equation (22.2). Therefore, \(F\{\xi\} \) is self-adjoint. \(Q.E.D.\)

This ingenious proof, which is due to I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, \textit{Proc. Roy. Soc. (London)}, \textbf{A244}, 17 (1958), relies only \(F\) being independent of \(\xi\), and on \(K\) and \(\delta W\) being symmetric functions of their arguments. It does not depend on any specific form of \(F\). In ideal MHD, \(F\) depends on \(V = \xi\) only through \(B_1\), defined by Faraday’s law and Ohm’s law; the explicit dependence on \(\xi\) integrates out through the definition \(Q = \nabla \times (\xi \times B_0)\). Note that, if the equilibrium is not stationary, so that \(V_0 \neq 0\), then the explicit dependence on \(\xi\) will remain, and \(F\) is no longer self-adjoint. Similarly, the so-called two-fluid extensions of Ohm’s law negate the convenient time integration that occurs in ideal MHD, and self-adjointness is also lost in this case.

From Equation (22.19), we have (with \(\xi = \eta\))

\[\delta W \{\xi, \xi\} = -K \left\{ \frac{1}{\rho_0} F\{\xi\}, \xi \right\} ,\]

\[= -\frac{1}{2} \int dV \xi \cdot F\{\xi\} .\]  \hspace{1cm} (22.21)

Therefore, if \(\xi \cdot F\{\xi\} > 0\) the displacement decreases the potential energy of the system and causes instability. This is consistent with our previous discussions.

In Section 19 we discussed some consequences of the self-adjointness of \(F\). These are:

1. The eigenfunctions \(-\omega_i^2\) are \textit{real}.  

\[\]
2. The eigenfunctions $\xi_k$ orthogonal.
3. The eigenfunctions $\xi_k$ form a complete set.

A further important consequence is that an energy principle exists. Substitute the eigenfunction expansion $\xi = \sum_j a_j \xi_j$ into the expression for the change in the potential energy, Equation (22.21), and use Equation (22.1) and the fact that $F$ is linear:

$$\delta W = \frac{1}{2} \sum_j \sum_k a_j a_k \omega_k^2 \int \rho_j \xi_j \cdot \xi_k dV,$$

$$= \frac{1}{2} \sum_j \sum_k a_j a_k \omega_k^2 \delta_{jk},$$

$$= \frac{1}{2} \sum_j a_j^2 \omega_j^2. \tag{22.22}$$

Therefore, $\delta W < 0$ if and only if there exists at least one unstable eigenmode (with $\omega_j^2 < 0$), i.e., $\delta W < 0$ is a necessary and sufficient condition for instability. Mathematically, this means that we can use a variational principle to test whether the system is unstable without having to solve the underlying boundary value problem. This is a very powerful tool for theoretical and computational analysis. However, it requires a discussion of the Calculus of Variations. We will defer this discussion until later in this course.