24. WAVES IN A UNIFORM MEDIUM: ARBITRARY ANGLE OF PROPAGATION

In Section 23 we showed that in an infinite, uniform medium the solutions of the ideal MHD wave equation could be decomposed into plane wave solutions \( \xi_k e^{i(k \cdot r + \omega t)} \) that satisfy

\[
\left\{ \begin{array}{c}
\omega^2 - (\hat{k} \cdot \hat{b})^2 V_A^2 \xi = \left[ (C_s^2 + V_A^2) (\hat{k} \cdot \xi) - V_A^2 (\xi \cdot \hat{b}) (\hat{k} \cdot \hat{b}) \right] \hat{k} \\
- V_A^2 (\hat{k} \cdot \xi) (\hat{k} \cdot \hat{b}) \hat{b}
\end{array} \right. 
\]  

(24.1)

There we examined several special cases of propagation both perpendicular and parallel to the magnetic field. Here we examine the more general case of propagation at an arbitrary angle \( \theta = \cos^{-1} (\hat{k} \cdot \hat{b}) \) to the magnetic field.

An Diversion on Homogenous Systems and Eigenvalue Problems

Equation (24.1) stands for three simultaneous homogenous equations in the three unknowns \( \xi_x, \xi_y, \) and \( \xi_z \). It is of the form

\[
(A - \omega^2 \mathbb{I}) \cdot \xi = 0 .
\]  

(24.2)

The obvious solution of Equation (24.2) is \( \xi = 0 \). We enquire as to what is needed to find non-trivial (i.e., \( \xi \neq 0 \)) solutions of this equation.

First consider the prototype equation \( ax = b \), where all variables are scalars. The solution is \( x = a / b \). If \( b = 0 \), it is clear that then only possible way to have \( x \neq 0 \) is to have \( a = 0 \). Then \( x \) can be anything, but it can always be written in terms of the solution \( x = 1 \).

Now consider the \( 2 \times 2 \) system

\[
\begin{align*}
\xi_1 &= a_1 x_1 + a_2 x_2 = b_1 , \\
\xi_2 &= a_{11} x_1 + a_{22} x_2 = b_2 .
\end{align*}
\]  

(24.3)

(24.4)

The solution is

\[
\begin{align*}
\xi_1 &= \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} , \\
x_1 &= \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} ,
\end{align*}
\]  

(24.5)

(24.6)
In analogy with the case \( ax = b \), if \( b_1 = b_2 = 0 \), the solution is \( x_1 = x_2 = 0 \) unless

\[
\begin{vmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \equiv \det \mathbf{A} = 0 \quad .
\tag{24.7}
\]

This means that equations (24.2) and (24.3) are no longer independent equations; one is a linear combination of the other. In that case we can determine the \textit{ratio} \( x_1 / x_2 \) from either (24.3) or (24.4); they must give the same result. From (24.3) we have \( x_1 / x_2 = -a_{12} / a_{11} \), and from (24.4) \( x_1 / x_2 = -a_{22} / a_{21} \). For these to be the same we require \( a_{21} / a_{11} = a_{22} / a_{21} \), which is identical to Equation (24.7). If the system were \( N \times N \), the vanishing of the determinant would ensure that only \( N - 1 \) of the equations are independent.

Now consider the eigenvalue problem

\[
(\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{x} = 0 \quad .
\tag{24.8}
\]

By the above argument, the only solution of Equation (24.8) is \( \mathbf{x} = 0 \), unless

\[
\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad .
\tag{24.9}
\]

This is called the \textit{characteristic equation}. The solution of this equation determines special values of \( \lambda \) for which the system of equations (24.8) are no longer independent. These are called \textit{eigenvalues}. Again consider the \( 2 \times 2 \) system

\[
\begin{pmatrix}
    a_{11} - \lambda & a_{12} \\
    a_{21} & a_{22} - \lambda
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix} = 0 \quad .
\tag{24.10}
\]

The characteristic equation is

\[
(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0 \quad .
\tag{24.11}
\]

It has two roots, \( \lambda_1 \) and \( \lambda_2 \). When \( \lambda = \lambda_1 \) or \( \lambda = \lambda_2 \), the two equations represented by (24.10) are no longer independent. For example, when \( \lambda = \lambda_1 \), \textit{either} of the equations

\[
(a_{11} - \lambda_1) x_1 + a_{12} x_2 = 0 \quad ,
\tag{24.12}
\]

\[
a_{21} x_1 + (a_{22} - \lambda_1) x_2 = 0 \quad ,
\tag{24.13}
\]

can be solved for the ratio \( x_1 / x_2 \). For these to yield the same result requires

\[
(a_{11} - \lambda_1)(a_{22} - \lambda_1) - a_{12}a_{21} = 0 \quad ,
\tag{24.15}
\]

which is just Equation (24.11). The same holds for \( \lambda = \lambda_2 \), but the ratio \( x_1 / x_2 \) will be different.

There are two vectors \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \), corresponding to \( \lambda_1 \) and \( \lambda_2 \), whose components \( x_1 \) and \( x_2 \) have the ratios \( x_1 / x_2 \) determined by this procedure. These are called \textit{eigenvectors}. Only the ratio of their components can be determined.

We now return to Equation (24.1). It is a \( 3 \times 3 \) system, so its characteristic equation will yield three roots for \( \omega^2 \). This results in 6 possible waves:
• Two shear waves, with $\omega = \pm \omega_1$;
• Two magneto-acoustic waves, with $\omega = \pm \omega_2$; and
• Two sound waves, with $\omega = \pm \omega_3$.

To be specific, we let $
\hat{b} = \hat{e}_z, \quad \mathbf{k} = k_{\parallel} \hat{e}_x + k_{\perp} \hat{e}_z, \quad \xi = \xi_x \hat{e}_x + \xi_y \hat{e}_y + \xi_z \hat{e}_z, \quad \text{and} \quad k^2 = k_{\parallel}^2 + k_{\perp}^2.\n$
Using this in Equation (24.1), we find
\begin{align}
\text{x-component:} & \quad \left( V_A^2 k^2 + C_S^2 k_{\perp}^2 \right) \xi_x + C_S^2 k_{\parallel} k_{\perp} \xi_z = \omega^2 \xi_x, \quad \text{(24.16)} \\
\text{y-component:} & \quad V_A^2 k_{\parallel}^2 \xi_y = \omega^2 \xi_y, \quad \text{(24.17)} \\
\text{z-component:} & \quad C_S^2 k_{\parallel} k_{\perp} \xi_x + C_S^2 k_{\parallel}^2 \xi_z = \omega^2 \xi_z. \quad \text{(24.18)}
\end{align}

Notice that the $y$-component decouples from the $x$- and $z$-components. This immediately gives the eigenvalue
\begin{equation}
\omega_0^2 = k_{\parallel}^2 V_A^2, \quad \text{(24.19)}
\end{equation}
or $\omega_0^2 = k^2 V_A^2 \cos^2 \theta$, and the eigenvector
\begin{equation}
\xi_0 = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad \text{(24.20)}
\end{equation}
i.e., $\xi_x = \xi_z = 0$, $\xi_y = 1$. This is the shear Alfvén wave found in Section 23. The polarization is shown in the figure.

The two remaining eigenvectors have $\xi_x \neq 0$, $\xi_z \neq 0$, and $\xi_y = 0$. The characteristic equation for the coupled $x$- and $z$-components (Equations (24.16) and (24.18)) is
\begin{equation}
\left( \frac{\omega}{k} \right)^4 - \left( C_S^2 + V_A^2 \right) \left( \frac{\omega}{k} \right)^2 + C_S^2 V_A^2 \cos^2 \theta = 0. \quad \text{(24.21)}
\end{equation}
The eigenvalues are
Estimates of these solutions can be made for the interesting special case of $C_s^2/V_A^2 = \Gamma \beta / 2 << 1$ (strong magnetic field). Then to lowest order in this parameter, the eigenvalue corresponding to the (+) sign is

$$\left( \frac{\omega}{k} \right)_{1,2}^2 = \frac{1}{2} \left( C_s^2 + V_A^2 \right) \left[ 1 \pm \sqrt{1 - \frac{4C_s^2V_A^2\cos^2\theta}{C_s^2 + V_A^2}} \right].$$

(24.22)

For $\theta = \pi/2$ ($\mathbf{k}$ perpendicular to $\hat{\mathbf{b}}$, or propagation across the field), this mode has phase velocity $\pm \sqrt{C_s^2 + V_A^2}$. This is just the \textit{magneto-acoustic wave} we found in Section 23. The polarization is shown in the figure.

For $\theta = 0$ (propagation parallel to the field), the phase velocity is $\pm V_A$. Note, however, that this is \textit{not} the shear Alfvén wave. The eigenvector corresponding to this frequency is found by substituting $\omega = \omega_i = k_iV_A$ (for $\theta = 0$) into Equations (24.16) and (24.18). The result is $(C_s^2 - V_A^2)\xi_z = 0$, so that $\xi_x = 1, \xi_z = 0$ is a non-trivial solution. The eigenvector is

$$\xi_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$  

(24.24)

This is different from the shear Alfvén wave eigenvector given by Equation (24.20). This is sometimes called the \textit{pseudo-mode}. (Why? I have no idea!) The polarization is shown in the figure.
The eigenvalue corresponding to the (-) sign in Equation (24.22) is, again in the strong field limit,
\[
\left( \frac{\omega}{k} \right)_2^2 = C_s^2 \cos^2 \theta,
\]
(24.25)
or \( \omega_2 = \pm C_s k \). The eigenvector is
\[
\xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]
(24.26)
This is called the slow branch of the magneto-acoustic wave. It becomes the sound wave for parallel propagation (when \( \theta = 0 \)).

The results for wave propagation in a uniform, infinite medium in the strong field limit are summarized in the phase velocity diagram, shown in the figure.

![Phase Velocity Diagram](image)

The magnetic field points in the \( z \)-direction (upward in the figure). The surfaces shown in the figure represent the tip of the phase velocity vector \( \mathbf{V}_p = (\omega / k) \mathbf{\hat{k}} \).

And that’s all there is for waves!