25. THE CALCULUS OF VARIATIONS AND THE IDEAL MHD ENERGY PRINCIPLE

In Section 22 we showed that the ideal MHD force operator is self-adjoint, and suggested that this allowed a formulation in which the stability of a system could be determined without solving a differential equation. This requires a little background in the calculus of variations. In the Section we begin this discussion, and formulate the ideal MHD energy principle.

We have seen that an equilibrium state is an extremum of the potential energy of the system; i.e., \( \partial W / \partial r_i = 0 \), where the \( r_i \) represent symbolically all possible local displacements of the system. Systems seek the lowest accessible state of potential energy. Therefore, if the extremum of the energy is a local minimum, the equilibrium is stable; if it is a local maximum, or an inflection point, the system is unstable (because other, lower energy states are accessible.)

In ordinary calculus, the extrema of a function \( F(x) \) are determined by the roots of \( F'(x) = 0 \). These roots are numbers (values of \( x \)) at which \( F \) has a local maximum, minimum, or inflection point.

Now consider the expression

\[
J \{ y \} = \int_{x_0}^{x_1} F(x, y, y') dx \ ,
\]

where \( y(x) \) is a function of \( x \), \( y' = dy/dx \), \( x_0 \), \( x_1 \), \( y(x_0) \), and \( y(x_1) \) are known numbers, and the functional form \( F(x, y, z) \) is given. The problem is to determine the function \( y(x) \) that “extremizes” \( J \{ y \} \). The formalism for solving this problem is called the calculus of variations. \( J \{ y \} \) is called a functional of \( y \), because its value depends on the particular functional form \( y(x) \). Once the functional form \( y(x) \) is known, \( J \{ y \} \) is just a number.

Consider a function \( \eta(x) \) defined in the interval \( x_0 \leq x \leq x_1 \), with \( \eta(x_0) = \eta(x_1) = 0 \), and let \( \varepsilon \) be a small number, i.e., \( \varepsilon << 1 \). Then the function \( \tilde{y} = y + \varepsilon \eta \) satisfies the same boundary conditions at \( x_0 \) and \( x_1 \) as \( y \). If \( y(x) \) is the extremizing function for \( J \{ y \} \), then for a local minimum \( \Phi(\varepsilon) \equiv J \{ y + \varepsilon \eta \} > J \{ y \} \), and for a local maximum \( \Phi(\varepsilon) \equiv J \{ y + \varepsilon \eta \} < J \{ y \} \), except when \( \varepsilon = 0 \). (We will not deal with inflection points.) The condition for an extremum is therefore \( d\Phi / d\varepsilon = 0 \) at \( \varepsilon = 0 \). More specifically,

\[
\Phi(\varepsilon) = \int_{x_0}^{x_1} F(x, y + \varepsilon \eta, y' + \varepsilon \eta') dx \ .
\]

Then

\[
\frac{d\Phi}{d\varepsilon} = \int_{x_0}^{x_1} \left[ \frac{\partial F}{\partial (y + \varepsilon \eta)} \frac{d}{d\varepsilon} (y + \varepsilon \eta) + \frac{\partial F}{\partial (y' + \varepsilon \eta')} \frac{d}{d\varepsilon} (y' + \varepsilon \eta') \right] dx \ ,
\]

(25.3)
or, at $\varepsilon = 0$,

$$
\frac{d\Phi}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_{x_0}^{x_1} \left[ \eta \frac{\partial F}{\partial y} + \eta y \frac{\partial F}{\partial y'} \right] dx = 0 
$$

(25.4)

must hold for all functions $\eta(x)$ satisfying $\eta(x_1) = \eta(x_2) = 0$. Now,

$$
\frac{d}{dx}\left( \eta \frac{\partial F}{\partial y'} \right) = \eta' \frac{\partial F}{\partial y'} + \eta \frac{d}{dx}\left( \frac{\partial F}{\partial y'} \right).
$$

(25.5)

Therefore

$$
0 = \int_{x_0}^{x_1} \left[ \eta \frac{\partial F}{\partial y} + \frac{d}{dx}\left( \eta \frac{\partial F}{\partial y'} \right) - \eta y \frac{d}{dx}\left( \frac{\partial F}{\partial y'} \right) \right] dx,
$$

$$
= \int_{x_0}^{x_1} \eta \left[ \frac{\partial F}{\partial y} - \frac{d}{dx}\left( \frac{\partial F}{\partial y'} \right) \right] dx + \eta \frac{\partial F}{\partial y'} \bigg|_{x_0}^{x_1},
$$

$$
= \int_{x_0}^{x_1} \eta \left[ \frac{\partial F}{\partial y} - \frac{d}{dx}\left( \frac{\partial F}{\partial y'} \right) \right] dx.
$$

(25.6)

The integrated term vanishes because of the boundary conditions on $\eta$.

Equation (25.6) must hold for all admissible functions $\eta(x)$. The \textit{fundamental lemma of the calculus of variations} is that this can only occur if

$$
\frac{\partial F}{\partial y} - \frac{d}{dx}\left( \frac{\partial F}{\partial y'} \right) = 0.
$$

(25.7)

This is called the \textit{Euler equation}. Using the notation $\frac{\partial F}{\partial y'} \equiv F_y'$, it is written as

$$
\frac{d}{dx} F_y - F_y = 0.
$$

(25.8)

or, expanding the first term,

$$
y'' F_{y y'} + y' F_{y y} + F_{y x} - F_y = 0.
$$

(25.9)

As seen explicitly from Equation (2.9), the Euler equation is a second order ODE for the function $y(x)$, with boundary conditions $y(x_0)$ and $y(x_1)$, that extremizes $J\{y\}$.

The function $\delta y = \varepsilon \eta$ is called the \textit{first variation of $y$}. The \textit{first variation of $J$} is defined as

$$
\delta J = J\{y + \varepsilon \eta\} - J\{y\},
$$

$$
= \Phi(\varepsilon) - \Phi(0),
$$

$$
\approx \frac{d\Phi}{d\varepsilon}\bigg|_{\varepsilon=0} \varepsilon,
$$

or from the Equations preceding (25.6),
The condition for an extremum is therefore \( \delta J = 0 \), i.e., the first variation of \( J \) must vanish. Note that, in general, the resulting expression depends explicitly on the boundary conditions through the last term in Equation (25.10). Boundary conditions for which

\[ \delta y \frac{\partial F}{\partial y'} \bigg|_{x_0} = 0 \]  

are called the natural boundary conditions for the problem \( \delta J = 0 \). This means that any solution of the variational problem

\[ \delta J = 0 = \int_{x_0}^{x_1} \delta y \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] dx \]

will automatically satisfy the natural boundary conditions.

Therefore, a necessary condition for \( y(x) \) to be an extremum of \( J \{y\} \) is the vanishing of the first variation of \( J \), \( \delta J = 0 \), for all \( y + \delta y \) that satisfy the boundary conditions.

Now, it is fair to ask how all this is related to ideal MHD. Recall that the ideal MHD wave equation is

\[ -\rho_0 \omega^2 \xi = F \{ \xi \} . \]  

(The reason for the \( \{ \ldots \} \) notation is not apparent; \( F \) is a functional of \( \xi \).) Dotting Equation (25.12) with \( \xi^* \) and integrating over all space, we can write

\[ \omega^2 \{ \xi \} = \frac{\delta W \{ \xi^*, \xi \}}{K \{ \xi^*, \xi \} } , \]  

where

\[ \delta W \{ \xi^*, \xi \} = -\frac{1}{2} \int \xi^* \cdot F \{ \xi \} dV , \]  

and

\[ K \{ \xi^*, \xi \} = \frac{1}{2} \int \xi^* \cdot \xi dV . \]

Therefore, \( \omega^2 \) is an integral functional of \( \xi \).

We now take the first variation of Equation (25.13). We will use the notation \( \Delta \) instead of \( \delta \) for the variation to avoid confusion with the standard notation \( \delta W \) for the potential energy. This procedure is facilitated by writing Equation (25.13) as

\[ K \omega^2 = -\frac{1}{2} \int \xi^* \cdot F \{ \xi \} dV . \]  

(25.17)
Taking the first variation,
\[ \Delta K \omega^2 + K \Delta \omega^2 = -\frac{1}{2} \int (\Delta \xi^* \cdot \mathbf{F}\{\xi\} + \xi^* \cdot \mathbf{F}\{\Delta \xi\}) dV \]  \hspace{1cm} (25.18)

Now,
\[ \Delta K = \int \rho_o \Delta \xi^* \cdot \xi dV \]  \hspace{1cm} (25.19)

and
\[ \int \xi^* \cdot \mathbf{F}\{\Delta \xi\} dV = \int \Delta \xi^* \cdot \mathbf{F}\{\xi\} dV \]  \hspace{1cm} (25.20)

since \(\mathbf{F}\) is self-adjoint. Then,
\[ \Delta \omega^2 = -\frac{1}{K} \int \Delta \xi^* \cdot \left( \mathbf{F}\{\xi\} + \rho_o \omega^2 \xi \right) dV \]  \hspace{1cm} (25.21)

Therefore, the condition for the vanishing of the first variation of \(\omega^2\), \(\Delta \omega^2 = 0\), is the same as requiring \(-\rho_o \omega^2 \xi = \mathbf{F}\{\xi\}\), i.e., the ideal MHD wave equation is the Euler equation for the variational problem \(\Delta \omega^2 = 0\). It is a consequence of \(\mathbf{F}\) being self-adjoint.

From Equations (25.13) and (25.14), we see that if there exists a displacement \(\xi(r)\) for which \(\delta W < 0\), then \(\omega^2 < 0\) and the system is unstable. This leads to the Ideal MHD Energy Principle: If \(\delta W \geq 0\) for all allowable displacements, then the system is stable. If \(\delta W < 0\) for any displacement, the system is unstable. These conditions can be shown to be both necessary and sufficient for determining the stability of an ideal MHD system.

Through the Energy Principle, the determination of stability is reduced to finding if there is a possible displacement for which \(\delta W < 0\). This is a great simplification. However, in order to apply it in specific situations we need to extend the energy principle to include contributions not just from the plasma, but also from the vacuum and the plasma/vacuum interface.

We can write the total potential energy of the perturbed system as
\[ \delta W = \delta W_F + \delta W_s + \delta W_v \]  \hspace{1cm} (25.22)

where \(\delta W_F = -(1/2) \int \xi^* \cdot \mathbf{F}\{\xi\} dV\) is the contribution from the plasma (fluid), \(\delta W_s\) is the contribution from the plasma/vacuum interface, and \(\delta W_v\) is the contribution from the vacuum. The explicit separation of \(\delta W\) as indicated in Equation (25.22) requires many integrations by parts. After "a tedious but straightforward calculation", it turns out that
\[ \delta W_F = \frac{1}{2} \int F dV \left[ \frac{Q_f^2}{\mu_0} - \xi_{\perp} \cdot (J \times Q) + \Gamma p_o |\nabla \cdot \xi|^2 + (\xi_{\perp} \cdot \nabla p_o) \nabla \cdot \xi_{\perp} \right] \]  \hspace{1cm} (25.23)

\[ \delta W_s = \frac{1}{2} \int S |\hat{n} \cdot \xi_{\perp}|^2 \hat{n} \cdot \nabla \left[ p + \frac{B^2}{2\mu_0} \right] \]  \hspace{1cm} (25.24)

and
\[ \delta W_v = \frac{1}{2} \int_V dV \frac{1}{\mu_0} \left| \nabla \times \hat{A} \right|^2, \]  

(25.25)

where \( F \) denotes the volume of the fluid, \( V \) denotes the volume of the vacuum, and \( S \) denotes the surface of the plasma/vacuum interface. We note that the vacuum contribution is always stabilizing, i.e., \( \delta W_v > 0 \).

As we have seen, the displacement and perturbed fields are subject to the conditions:

\[ \hat{n} \times \hat{A} = 0 \]  

(25.26)

for the vacuum solution, or

\[ \hat{n} \cdot \xi = 0 \]  

(25.27)

for the plasma solution, at the outer conducting boundary; and

\[ \hat{n} \times \hat{A} = -(\hat{n} \cdot \xi) \hat{B}_0 \]  

(25.28)

and

\[ \Gamma p_0 \nabla \cdot \xi + \frac{1}{\mu_0} (\xi \cdot \nabla \hat{B}_0 + \hat{B}_1) \cdot \hat{B}_0 = \frac{1}{\mu_0} (\xi \cdot \nabla \hat{B}_0 + \hat{B}_1) \cdot \hat{B}_0, \]  

(25.29)

at the plasma/vacuum interface.

Since we only need to show that there is some displacement that makes \( \delta W < 0 \) and satisfies the boundary conditions, the usual procedure is to guess a functional form for \( \xi \), called a trial function, substitute it into the energy principle, and compute \( \delta W \). (This procedure is not as random as it appears because of the Rayleigh-Ritz technique, which will be discussed in a later Section.) Of course, the trial function must satisfy the conditions of Equations (25.26) – (25.29). It turns out that the pressure balance boundary condition, Equation (22.29), is difficult to enforce. However, it can be shown (but won’t be here) that for every displacement \( \xi \) that does not satisfy pressure balance on \( S \) but makes \( \delta W < 0 \), there is a “neighboring” displacement \( \tilde{\xi} \) that satisfies pressure balance whose energy \( \delta \tilde{W} \) differs from \( \delta W \) by an arbitrarily small amount. Therefore, it is not necessary to chose a trial function that satisfies the pressure balance condition. If a system is unstable without satisfying pressure balance, it will also be unstable if pressure balance is enforced.

This leads to what is called the Extended Energy Principle: A necessary and sufficient condition for instability is that one can find a \( \xi \) and \( \hat{A} \) that satisfy \( \hat{n} \times \hat{A} = -(\hat{n} \cdot \xi) \hat{B}_0 \) at the fluid/vacuum interface, and have either \( \hat{n} \times \hat{A} = 0 \) or \( \hat{n} \cdot \xi = 0 \) at conducting boundaries, which makes \( \delta W < 0 \).

The Extended Energy Principle forms the basis for much of linear stability theory, especially for fusion plasmas.