28. THE GRAVITATIONAL INTERCHANGE MODE, OR g-MODE

We now consider the case where the magneto-fluid system is subject to a gravitational force $\mathbf{F}_g = \rho g$, where $g$ is a constant gravitational acceleration vector. In a system with straight field lines, the equilibrium condition is

$$\nabla \left( p + \frac{B^2}{2\mu_0} \right) = \rho g . \quad (28.1)$$

Recall that in a system with curved field lines, but no gravity, the equilibrium condition is

$$\nabla \left( p + \frac{B^2}{2\mu_0} \right) = \frac{B^2}{\mu_0} \kappa , \quad (28.2)$$

where $\kappa = \hat{b} \cdot \nabla \hat{b}$ is the field line curvature. Therefore, by using gravity as a proxy force, it is possible to study the stability properties of systems with curved field lines in Cartesian geometry with straight field lines. This is a great simplification. This accounts for both the importance of the gravitational problem in MHD, and the richness of its solutions. The study of the stability properties of the equilibrium given by Equation (28.1), the so-called g-mode, or gravitational interchange problem, is one of the most important problems in MHD.

Since $\mathbf{F}_g = \rho g$, the gravitational force will make an additional contribution to the perturbed potential energy of

$$\delta W_g = -\frac{1}{2} \int dV \xi^* \cdot \mathbf{F}_g \{ \xi \} . \quad (28.3)$$

The perturbed gravitational force is $\mathbf{F}_g = \rho g$. We have previously shown that the perturbed density is related to the displacement by

$$\rho_i = -\xi \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \xi , \quad (28.4)$$

so that

$$\delta W_g = \frac{1}{2} \int dV \left( \xi^* \cdot g \right) \left( \xi \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \xi \right) . \quad (28.5)$$

The full expression for $\delta W$ including gravity is

$$\delta W = \frac{1}{2} \int dV \left\{ \frac{|\mathbf{Q}|^2}{\mu_0} - \xi^* \cdot \mathbf{J} + \Gamma \rho_0 \left| \nabla \cdot \xi \right|^2 + \left( \xi^* \cdot \nabla \rho_0 \right) \nabla \cdot \xi + \left( \xi^* \cdot g \right) \left( \xi \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \xi \right) \right\} \quad (28.6)$$

where $\mathbf{Q} = \nabla \times (\xi \times \mathbf{B})$.

We now consider the equilibrium of Equation (28.1) in Cartesian geometry. Solid walls are at $x = \pm L$, $\mathbf{B} = B(x)\hat{e}_y$, and $g = g\hat{e}_x$. The current is
\[ J_z = \frac{1}{\mu_0} \frac{dB}{dx}, \quad (28.7) \]

and the force balance condition, Equation (28.1), is
\[ \frac{dp_0}{dx} + \frac{1}{2\mu_0} \frac{dB^2}{dx} = \rho_0 g \quad . \quad (28.8) \]

The configuration is shown in the figure.

We now specialize to the case \( \xi = \xi(x, z) \), so that the displacement is independent of \( z \). Note that now \( \xi \) is real, so \( \xi^* = \xi \). We substitute this ansatz into Equation (28.6). After considerable algebra, the result is
\[ \delta W = \int dV \left[ \left( \frac{B^2}{\mu_0} + \Gamma p_0 \right) (\nabla \cdot \xi)^2 + 2\rho_0 g \xi_x \nabla \cdot \xi + g \frac{d\rho_0}{dx} \xi_x^2 \right] . \quad (28.9) \]

First, we note that the system is stable \( (\delta W > 0) \) if \( g = 0 \).

Second, if \( \nabla \cdot \xi = 0 \), stability requires
\[ \int g \frac{d\rho_0}{dx} \xi_x^2 > 0 . \quad (28.10) \]

Suppose that \( gd\rho_0 / dx < 0 \) at a single point \( x_0 \). Then we could choose a trial function \( \xi_x \) that is zero everywhere except in a very small region around \( x_0 \), as shown in the figure.
Then $\delta W < 0$ and the system is unstable. But the point $x_0$ is arbitrary, so the condition for stability $g d\rho_0 / dx > 0$ must hold pointwise (i.e., at all points in $-L < x < L$), for if it is not satisfied at a single point, then $\delta W < 0$ and the system is unstable. This is called the Rayleigh-Taylor instability. Stable and unstable situations for incompressible perturbations are sketched in the figure.

Third, $\nabla \cdot \xi \neq 0$ is not always stabilizing when $\nabla \rho_0 \neq 0$; incompressible displacements are not always the most unstable. In this case, if $\xi_x \nabla \cdot \xi > 0$, then the system is unstable if $g < 0$; if $\xi_x \nabla \cdot \xi < 0$, then the system is unstable if $g > 0$. Note that these conclusions are true even if $d\rho_0 / dx = 0$. Consider the total pressure profile shown in the figure on the left. Let the fluid element at $x$ be displaced downward to the new position $x'$, as shown in the figure on the right.

The pressure at $x'$ is greater than the pressure at $x$. During the displacement the fluid element will therefore be compressed ($\nabla \cdot \xi < 0$), its density will increase, and it will find itself heavier than its surroundings. It will continue to fall. Similarly, upwardly displaced elements will expand, find themselves lighter than their surroundings, and continue to climb. This is called buoyancy. It is an instability that depends on compressibility. It is often called the Parker instability.
In the \( g \)-mode, the system lowers its energy by interchanging fluid elements. Instabilities of this type are called, not surprisingly, \textit{interchange modes}. The \( g \)-mode is often described as the instability of a heavier fluid being supported by a lighter one. This is the case when the fluid is incompressible, and is its primary manifestation in magnetic confinement devices, such as tokamaks, where incompressibility is enforced by a strong magnetic field (as in Reduced MHD). However, we have seen that, when the fluid is compressible, instability can occur even when there is no density gradient. This is often the primary manifestation of this mode in astrophysical settings.

We now return to the Energy Principle, Equation (28.9). Notice that the integrand is a quadratic form in the independent variables \( \nabla \cdot \xi \) and \( \xi_x \), i.e., it can be written as

\[
\sum_{i,j=1}^{2} x_i a_{ij} x_j = \left( \frac{B^2}{\mu_0} + \Gamma p_0 \right) (\nabla \cdot \xi)^2 + \rho_0 g (\nabla \cdot \xi) \xi_x + \rho_0 g \xi_x (\nabla \cdot \xi) + g \frac{d\rho_0}{dx} \xi_x^2 ,
\]

where \( x_1 = \nabla \cdot \xi \) and \( x_2 = \xi_x \). If this is positive, the system is stable. There is a theorem stating that a necessary and sufficient condition for a quadratic form to be positive, i.e.,

\[
Q = x^T \cdot A \cdot x > 0 ,
\]

is that the determinant of all the principal minors of \( A \) be greater than zero. That is,

\[
\det A_1 = a_{11} > 0 ,
\]

\[
\det A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 ,
\]

\[
\det A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0 ,
\]

etc, etc., must hold simultaneously. Equation (20.10) is a \( 2 \times 2 \) system, with

\[
a_{11} = \frac{B^2}{\mu_0} + \Gamma p_0 , \quad a_{12} = \rho_0 g ,
\]

\[
a_{21} = \rho_0 g , \quad a_{22} = g \frac{d\rho_0}{dx} .
\]

The fist principal determinant is

\[
a_{11} = \frac{B^2}{\mu_0} + \Gamma p_0 > 0 ,
\]

which is always satisfied. Requiring the second principal determinant (in this case, the determinant of \( a_{ij} \)) to be positive yields

\[
\left( \frac{B^2}{\mu_0} + \Gamma p \right) \left( g \frac{d\rho_0}{dx} \right) - \left( \rho_0 g \right)^2 > 0 ,
\]
or
\[
g \frac{d\rho_0}{dx} > \frac{g^2}{C^2} > 0 ,
\tag{28.12}
\]
where \( C^2 = C_s^2 + V_A^2 \). Equation (28.12) is the condition for stability of the \( g \)-mode, including compressibility. As concluded in the discussion following Equation (28.10), it must hold at every point in \( x \). If it is violated at any point, the system is unstable.

There are two points:

1. The system is unstable if \( g \) and \( d\rho_0/dx \) have opposite signs. This is the same conclusion as in the incompressible case (the Rayleigh-Taylor instability).

2. Even if \( g \) and \( d\rho_0/dx \) have the same sign, the system will be unstable unless \( g (d\rho_0/dx) > \rho_0 g^2/C^2 \). Because of compressibility, instability can occur even when a heavy fluid supports a light fluid. The drive for the Parker instability can overcome the stabilization of the Rayleigh-Taylor instability (which occurs because of the alignment of the density gradient with gravity). This situation is sketched below.

We conclude by estimating the growth rate with the Rayleigh-Ritz technique. For simplicity, we only consider the incompressible case, \( \nabla \cdot \nabla = 0 \), i.e., the Rayleigh-Taylor instability. (Yes, it is the same Lord Rayleigh.) We assume an exponential density profile \( \rho_0 = \bar{\rho} e^{x/L_s} \). The perturbed potential energy is
\[
\delta W = \frac{1}{2} \int dV g \frac{d\rho_0}{dx} \frac{g^2}{C^2} . \tag{28.13}
\]
We choose the trial function
\[ \xi_x = \xi_0 e^{ikz} \cos \frac{\pi x}{2L}, \] \hspace{1cm} (28.14)

which is periodic in \( z \) and satisfies the boundary conditions \( \xi_x(-L, z) = \xi_x(L, z) = 0 \), as required. Since \( \nabla \cdot \xi_x = 0 \), the \( z \)-component of the displacement satisfies \( \partial \xi_z / \partial z = -\partial \xi_x / \partial x \). Using Equation (28.14) and integrating,

\[ \xi_z = \frac{-i\pi \xi_0}{2kL} e^{ikz} \sin \frac{\pi x}{2L}. \] \hspace{1cm} (28.15)

The potential energy is

\[ \delta W = \frac{\rho_0 g \xi_z^2}{2L_n} \int_{-L}^{L} \cos^2 \frac{\pi x}{2L} \, dx, \]

\[ = \frac{\pi \rho_0 g \xi_0^2}{4L_n}, \] \hspace{1cm} (28.16)

and the perturbed kinetic energy is

\[ K = \frac{1}{2} \rho_0 \int_{-L}^{L} (\xi_x^2 + \xi_z^2) \, dx, \]

\[ = \frac{\pi \rho_0 \xi_0^2}{4} \left( 1 + \frac{\pi^2}{4k^2L^2} \right). \] \hspace{1cm} (28.17)

The estimated growth rate is

\[ \omega^2 = \frac{\delta W}{K}, \]

\[ = \frac{g}{L_n} \frac{1}{1 + \pi^2 / 4k^2L^2}. \] \hspace{1cm} (28.19)

The system is unstable if \( g / L_n < 0 \), as expected. For long wavelengths, \( kL \ll 1 \), we have

\[ \gamma^2 \approx \frac{4}{\pi^2} (kL)^2, \] \hspace{1cm} (28.20)

and for short wavelengths, \( kL \gg 1 \),

\[ \gamma^2 \approx g / L_n, \] \hspace{1cm} (28.21)

where \( \gamma^2 = -\omega^2 \) is the square of the growth rate. When the wavelength is long, the mode can “feel” the presence of the conducting wall, and the growth rate is reduced. However, when the wavelength is short, the mode can evolve as though the wall were absent, and the growth rate is independent of \( k \). The full dependence of the growth rate on wave number is given by Equation (18.19). This is sketched in the figure.