29. COMMENTS ON THE ENERGY PRINCIPLE AND THE MINIMIZING EIGENFUNCTION

In this Section we make some general comments on the MHD Energy Principle and the process of minimization. These remarks are quite general. Some of them require a considerable amount of vector algebra and integration by parts. Most of these details have been omitted for clarity of presentation.

After a formidable calculation, the ideal MHD Energy Principle (excluding gravity) can be written as

$$
\delta W = \frac{1}{2} \int dV \left\{ \frac{|Q_{\perp}|^2}{\mu_0} + \frac{B^2}{\mu_0} \left| \nabla \cdot \delta \xi + 2 \xi_{\perp} \cdot \delta \mathbf{k} \right|^2 + \Gamma p_0 |\nabla \cdot \xi|^2 + 2 (\delta \xi_{\perp} \cdot \nabla p_0) (\mathbf{k} \cdot \nabla \xi_{\perp}) - B \left( \delta \xi_{\perp} \times \mathbf{b} \right) \cdot Q_{\perp} \right\},
$$

(29.1)

where

$$
Q = Q_{\perp} + Q_{\parallel} \mathbf{b},
$$

(29.2)

$$
Q_{\parallel} = -B (\nabla \cdot \xi_{\perp} + 2 \xi_{\perp} \cdot \mathbf{k}) + \frac{\mu_0}{B} \xi_{\perp} \cdot \nabla p_0,
$$

(29.3)

$$
\mathbf{J} = J_{\perp} + J_{\parallel} \mathbf{b},
$$

(29.4)

$$
\xi = \xi_{\perp} + \xi_{\parallel} \mathbf{b},
$$

(29.5)

and

$$
\mathbf{k} = \mathbf{b} \cdot \nabla \mathbf{b}.
$$

(29.6)

Instability occurs if \( \delta W < 0 \).

We remark on the terms in Equation (29.1).

1. \( |Q_{\perp}|^2 / \mu_0 > 0 \) is stabilizing. This is the energy required to bend the field lines. It gives rise to the shear Alfvén wave.

2. \( B^2 |\nabla \cdot \delta \xi + 2 \xi_{\perp} \cdot \delta \mathbf{k}|^2 / \mu_0 > 0 \) is stabilizing. It is the energy required to compress the magnetic field. It gives rise to magneto-acoustic waves.

3. \( \Gamma p_0 |\nabla \cdot \xi|^2 > 0 \) is stabilizing. It is the energy required to compress the fluid. It gives rise to sound waves.

4. \( -2 (\delta \xi_{\perp} \cdot \nabla p_0) (\mathbf{k} \cdot \nabla \xi_{\perp}) \) can be either stabilizing or destabilizing. When negative, it gives rise to pressure driven instabilities. Since \( \nabla p = \mathbf{J}_{\perp} \times \mathbf{B} \), it contains the effects of perpendicular current.
5. $-J_i (\xi_\perp \times \hat{b}) \cdot Q_\perp$ can be either stabilizing or destabilizing. When negative, it can give rise to current driven instabilities, or, more accurately, instabilities driven by the parallel current.

Note that $\xi_{||}$ enters $\delta W$ only through the term $\nabla \cdot \xi$. It is therefore possible to minimize $\delta W$ once and for all with respect to $\xi_{||}$. We will then be left to deal only with $\delta W \{\xi_\perp, \xi_{||}\}$. This minimization is carried out by letting $\xi_{||} \rightarrow \xi_{||} + \Delta \xi_{||}$ in Equation (29.1), where $\Delta \xi_{||}$ is the variation of $\xi$. (The $\Delta$ notation is used to avoid confusion with $\delta W$.) Assuming that $\hat{b} \cdot \hat{n} = 0$ on the boundary, and after much algebra, we finally arrive at an expression for the variation of $\delta W$ as

$$\Delta (\delta W) = 0 = -\text{Re} \int dV \Delta \xi_{||} \hat{b} \cdot \nabla (\Gamma \rho_0 \nabla \cdot \xi) .$$

(29.7)

Since this must hold for arbitrary $\Delta \xi_{||}$, we must require

$$\hat{b} \cdot \nabla (\Gamma \rho_0 \nabla \cdot \xi) = 0 .$$

(29.8)

Any $\xi$ that satisfies Equation (29.8) minimizes $\delta W$ with respect to $\xi_{||}$. Now

$$\hat{b} \cdot \nabla (\rho_0 \nabla \cdot \xi) = \rho_0 \hat{b} \cdot \nabla (\nabla \cdot \xi) + \nabla \cdot \xi \hat{b} \cdot \nabla \rho_0 .$$

(29.9)

But in equilibrium, $\hat{b} \cdot \nabla \rho_0 = 0$, so that Equation (29.8) becomes

$$\hat{b} \cdot \nabla (\nabla \cdot \xi) = 0 .$$

(29.10)

This is the final form of the minimizing condition.

Equation (29.10) has a form similar to the homogeneous algebraic equation $Ax = 0$, which we discussed in a previous Section. We know that if $A \neq 0$, then the only solution is $x = 0$. The condition $A \neq 0$ is equivalent to saying that $A$ is invertible, i.e., $A^{-1} \equiv 1/A$ exists. Conversely, if $A = 0$, $A$ is not invertible, and solutions $x \neq 0$ are possible.

By analogy, the properties of the constraint (29.10) depend on whether the operator $\hat{b} \cdot \nabla$ is invertible or not. Suppose it invertible everywhere. Then the solution of Equation (29.10) is $\nabla \cdot \xi = 0$; this is the minimizing condition. In light of Equation (29.5),

$$\nabla \cdot (\xi_\perp + \xi_{||} \hat{b}) = 0 .$$

(29.11)

With $\hat{b} = B / B$ and $\nabla \cdot B = 0$, this can be written as

$$B \cdot \nabla \left( \frac{\xi_{||}}{B} \right) = -\nabla \cdot \xi_\perp .$$

(29.12)

Choosing $\xi_{||}$ to satisfy Equation (29.12) will minimize $\delta W$ if $\hat{b} \cdot \nabla$ is invertible everywhere.
Now suppose that $\mathbf{b} \cdot \nabla$ is invertible everywhere except at some isolated locations where $\mathbf{b} \cdot \nabla = 0$. Then for $\nabla \cdot \xi_\perp$ to remain finite, $\xi_\parallel / B$ must go to infinity at these locations, i.e., $\xi_\parallel$ is singular there. This is not an acceptable trial function. Let the location of the singularity be $x_0$, and let

$$\xi = \tilde{\xi} + \epsilon \eta,$$

where $\eta(x)$ is zero everywhere except in a small region of size $\epsilon$ around $x = x_0$, and $\epsilon << 1$. We assume that $\xi$ is the “real” minimizing trial function that satisfies Equation (29.12) (i.e., has $\nabla \cdot \xi = 0$ everywhere and $\xi_\parallel$ singular at $x = x_0$), and that $\tilde{\xi}$ is some “neighboring” trial function (with $\nabla \cdot \tilde{\xi} \neq 0$ and $\xi_\parallel$ well behaved at $x = x_0$). Then $\nabla \cdot \xi = \nabla \cdot \tilde{\xi} + \epsilon \nabla \cdot \eta$, so that $\nabla \cdot \xi$ and $\nabla \cdot \tilde{\xi}$ can be chosen to be equal to each other (and therefore zero) away from $x_0$, and differ by an arbitrarily small amount near $x_0$. Then

$$\delta W \{\xi, \xi\} = \delta W \{\tilde{\xi} + \epsilon \eta, \tilde{\xi} + \epsilon \eta\},$$

$$= \delta \tilde{W} + O(\epsilon),$$

where $\delta \tilde{W} = \delta W \{\tilde{\xi}, \tilde{\xi}\}$ is a “neighboring” $\delta W$ computed with a trial function that has $\tilde{\xi}_\parallel$ well behaved at $x_0$. Now suppose $\delta \tilde{W} < 0$, i.e., we find instability with the trial function $\tilde{\xi}$. Then we can choose $\epsilon$ so small that $\delta W$ (the “real” $\delta W$) is also negative. We therefore conclude that for every well behaved trial function $\tilde{\xi}$ that makes $\delta \tilde{W} < 0$, there is a “neighboring” trial function $\xi$ (with $\xi_\parallel$ singular at $x_0$) whose potential energy $\delta W$ differs from $\delta \tilde{W}$ by an arbitrarily small amount, and which satisfies $\nabla \cdot \xi = 0$.

1. The procedure when $\mathbf{b} \cdot \nabla \neq 0$ at $x = x_0$ is as follows:
2. Choose a trial function $\tilde{\xi}$ that is well behaved at $x_0$.
3. Compute $\delta \tilde{W}$.
4. If $\delta \tilde{W} < 0$, the system is unstable.
5. It will also be unstable with the singular trial function $\xi$.

Therefore, any conclusions drawn about instability with a non-singular trial function will be valid.

Now suppose that $\mathbf{b} \cdot \nabla = 0$ everywhere. This is the case for the $g$-mode (with $\mathbf{B} = B(x)\hat{e}_x$ and $\nabla = \hat{e}_x \partial_x + \hat{e}_y \partial_y$) and for axially symmetric perturbations to a pure Z-pinch (with $\mathbf{B} = B_0(r)\hat{e}_z$ and $\nabla = \hat{e}_r \partial_r + \hat{e}_z \partial_z$). Then $\nabla \cdot \xi = \nabla \cdot \xi_\perp$, i.e., $\xi_\parallel$ does not appear in $\delta W$ and the term $\Gamma p_0 |\nabla \cdot \xi_\perp|^2$ must be retained (as was seen in the stability analysis of the $g$-mode). In this case, $\nabla \cdot \tilde{\xi} = 0$ is not the most unstable displacement.
A further special case occurs if the field lines are all closed. Then the periodicity constraint $\xi(l) = \xi(l + L)$, where $L$ the length of a field line and $l$ is the distance along a field line, must be imposed on each field line. It can be shown (but not here!), that this implies the condition

$$\int \Gamma p|\nabla \cdot \xi|^2 dV = \int \Gamma p |(\nabla \cdot \xi_\perp)|^2 dV,$$

where

$$\langle f \rangle \equiv \frac{\int dl f}{\int dl B}$$

denotes the average taken along a field line. Again, the term $\Gamma p|\nabla \cdot \xi_\perp|^2$ must be included in the analysis.

We remark on the significance of the condition $\hat{b} \cdot \nabla = 0$, which occurs at locations $x = x_0$. In the analysis of axisymmetric equilibria, the $x_0$ correspond to flux surfaces. These special surfaces are called singular surfaces. If we use the analogy $\nabla \rightarrow i\mathbf{k}$, then the singularity condition is $\mathbf{k} \cdot \mathbf{B} = 0$, i.e., on these surfaces the wavefronts of the perturbation are aligned with the magnetic field, so that the perturbation does not bend the field lines. We have already seen that this is destabilizing. Singular surfaces play a special role in the stability analysis of axisymmetric systems.

We will now consider several examples of the role played by the issues raised in this Section in determining the stability properties of some specific equilibria.