30. EXAMPLES OF THE APPLICATION OF THE ENERGY PRINCIPLE TO CYLINDRICAL EQUILIBRIA

We now use the Energy Principle to analyze the stability properties of the cylindrical \( \theta \)-pinch, the \( Z \)-pinch, and the General Screw Pinch. Since these equilibria depend only on the radial coordinate \( r \), are periodic in the \( \theta \)- and \( z \)-coordinates, and the fields are independent of the periodic coordinates, the displacement can be written as in terms of the Fourier decomposition

\[
\xi(r) = \xi(r) e^{i(m\theta + kz)}.
\]  

(30.1)

Since the equilibria have no radial component of the magnetic field,

\[
B \cdot \nabla = \frac{im}{r} B_\theta + ik B_z.
\]  

(30.2)

In Equation (30.1), \( m \) is an integer in the range \(-\infty \leq m \leq \infty\). It is called the \textit{poloidal mode number}. If the system is infinitely long, \( k \) is a continuous variable. However, it is quantized if the cylinder has finite length. For example, if a torus with circular cross section \( a \) and major radius \( R \) (has \textit{aspect ratio} \( R/a \)) is cut and straightened into a cylinder of length \( L = 2\pi R \), then periodicity requires \( k = n / R \), where \( n \) is an integer in the range \(-\infty \leq n \leq \infty\). It is called the \textit{axial, or toroidal, mode number}.

Typical displacements of the plasma column for different values of \( m \) and \( k \) are shown in the figure.
The mode with $m = 0, k \neq 0$ is colorfully called the “sausage mode". Since $m = 0$, the displacement is azimuthally symmetric (i.e., independent of $\theta$). The mode with $m = 1, k = 0$ is just a shift of the column with respect to the axis. The mode with $m = 1, k \neq 0$ is called a “kink mode”. It distorts the column helically.

We now consider the three cylindrical equilibria.

The $\theta$-Pinch

The equilibrium for the $\theta$-pinch is

$$ p(r) + \frac{B_z^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}, \quad (30.3) $$

where $B_0$ is the magnetic field strength outside the fluid. It is produced externally.

Since $B_0 = 0$, Equation (30.2) becomes $\mathbf{B} \cdot \nabla = i k B_z$, so that this operator can be inverted as long as $k \neq 0$. In that case the minimizing condition for $\delta W$ is $\nabla \cdot \xi = 0$, or

$$ \frac{1}{r} \frac{d}{dr} (r \xi_r) + \frac{im}{r} \xi_\theta + ik \xi_z = 0. \quad (30.4) $$

We can therefore use Equation (30.4) to eliminate $\xi_\parallel = \xi_z$ from $\delta W$ according to

$$ \xi_z = \frac{i}{kr} \left[ (r \xi_r)' + im \xi_\theta \right], \quad (30.5) $$

where $(...)'$ denotes differentiation with respect to $r$. This is valid as long as $k \neq 0$. In this case we have

$$ \mathbf{Q}_\perp = ik B_z \xi_\perp = ik B_z \left( \xi_r \mathbf{e}_r + \xi_\theta \mathbf{e}_\theta \right), \quad (30.6) $$

$$ \nabla \cdot \xi_\perp = \frac{1}{r} (r \xi_r)' + \frac{im \xi_\theta}{r}, \quad (30.7) $$

$$ \kappa = \mathbf{b} \cdot \nabla \mathbf{b} = 0, \quad (30.8) $$

and

$$ J_\parallel = 0. \quad (30.9) $$
In cylindrical geometry the potential energy per unit length is

$$\frac{\delta W}{L} = \frac{\pi}{\mu_0} \int_0^a W(r)rdr \quad .$$

(30.10)

For the case $k \neq 0$, using Equations (30.5) – (30.9),

$$W(r) = B_z^2 \left[ k^2 \left( |\xi_r|^2 + |\xi_r'|^2 \right) + \frac{1}{r^2} \left( |r\xi_r'|^2 \right)^2 + \frac{m^2}{r^2} |\xi_0|^2 + \frac{im}{r^2} (r\xi_r')' \xi_0^* - \frac{im}{r^2} (r\xi_r) (r\xi_r')^* \right] .$$

(30.11)

Then after a considerable amount of algebra, Equation (30.10) can be re-written as

$$\frac{\delta W}{L} = \frac{\pi}{\mu_0} \int_0^a rdrB_z^2 \left[ k_0^2 |\xi_0|^2 - \frac{im}{k_0} \left( r\xi_r' \right)^2 + \frac{k^2}{k_0^2 r^2} \left( \left( r\xi_r' \right)^2 + r^2 k_0^2 |\xi_0|^2 \right) \right] ,$$

(30.12)

where $k_0^2 = m^2 / r^2 + k^2$. We note that $\xi_0$ appears only in the first term in the integrand. Setting this term to zero, we find that the minimizing trial function must have

$$\xi_0 = \frac{im}{m^2 + k^2 r^2} (r\xi_r')' \quad ,$$

(30.13)

so that Equation (30.12) becomes

$$\frac{\delta W}{L} = \frac{\pi}{\mu_0} \int_0^a rdr \frac{k^2 B_z^2}{m^2 + k^2 r^2} \left[ \left( r\xi_r' \right)^2 + (m^2 + k^2 r^2) |\xi_0|^2 \right] .$$

(30.14)

We can draw two conclusions from Equation (30.14):

- $\delta W > 0$ for $k^2 \neq 0$, so that the $\theta$-pinch is stable for all finite $k$.
- $\delta W \to 0$ for $k^2 \to \infty$, so that the stability becomes marginal for very long wavelengths.

The MHD stability of the $\theta$-pinch is explained as follows:

- $J_\parallel = 0$ ( $J = J_\theta \hat{\theta}$, $B = B_r \hat{r}$ ), so that there are no current driven modes.
- $\kappa = 0$ (no field line curvature), so $(\xi_\perp \cdot \nabla p_0) (k \cdot \xi_\perp) = 0$ and there are no pressure driven modes.
- Therefore, there is no MHD source of “free energy” to drive instability.

However, real $\theta$-pinch have finite length and therefore have field line curvature, so this can drive instability in the laboratory. This is the reason we have not considered the case $k = 0$ here. Also, there are also several non-MHD instability drives that have rendered this concept problematical for magnetic confinement fusion. Hence, $\theta$-pinches are not a “mainstream” concept.

*The Z-Pinch*
For the Z-pinch, \( \mathbf{B} = B_\theta(r) \hat{\mathbf{e}}_\theta \) and \( \mathbf{J} = J_z(r) \hat{\mathbf{e}}_z \), and the equilibrium condition is

\[
\frac{dp_0}{dr} + \frac{B_\theta}{\mu_0 r} \frac{d}{dr} (r B_\theta) = 0 \quad .
\] (30.15)

Therefore \( J_\parallel = 0 \) and \( \xi_\parallel = \xi_\theta \). For \( m \neq 0 \), the operator \( \mathbf{B} \cdot \nabla = imB_\theta / r \) is well behaved and can be inverted. In the case the minimizing condition is \( \nabla \cdot \mathbf{\xi} = 0 \), so that

\[
\xi_\theta = \frac{i}{m} \left[ (r \xi_r)' + ik \xi_z \right] ,
\] (30.16)

which is valid as long as \( m \neq 0 \). If \( m = 0 \), then \( \mathbf{B} \cdot \nabla = 0 \) everywhere and we must consider \( \nabla \cdot \mathbf{\xi}_\perp \) in the minimization.

When \( m \neq 0 \), we have

\[
Q_\perp = \frac{imB_\theta}{r} (\xi_r \hat{\mathbf{e}}_r + \xi_\theta \hat{\mathbf{e}}_\theta) \quad ,
\] (30.17)

\[
\nabla \cdot \mathbf{\xi}_\perp + 2 \xi_\perp \cdot \mathbf{\kappa} = r \left( \frac{\xi_r}{r} \right)' + ik \xi_z ,
\] (30.18)

\[
\mathbf{\kappa} = - \frac{\hat{\mathbf{e}}_r}{r} ,
\] (30.19)

and

\[
J_\parallel = 0 \quad .
\] (30.20)

Again, after much algebra, we find

\[
\frac{\delta W}{L} = \frac{\pi}{\mu_0} \int_0^a dr \left\{ \left( 2 \mu_0 r p' + m^2 B_\theta^2 \right) \left[ \frac{\xi_r}{r} \right]' \left[ \frac{\xi_r}{r} \right]' + \frac{m^2 r^2 B_\theta^2}{m^2 + r^2 k^2} \left( \frac{\xi_r}{r} \right)' \right\} \quad .
\] (30.21)

The last term is minimized for \( k^2 \to \infty \), so the stability of the Z-pinch is determined by the sign of

\[
\frac{\delta W}{L} = \frac{\pi}{\mu_0} \int_0^a dr \left[ 2 \mu_0 r p' + m^2 B_\theta^2 \right] \left[ \frac{\xi_r}{r} \right]' \quad .
\] (30.22)

Suppose the integrand of Equation (30.22) is negative in some interval \( r_0 < r < r_1 \). Then we can choose the trial function such that \( \xi_r \neq 0 \) inside this interval, and \( \xi_r = 0 \) outside it. Since this interval is arbitrary, we conclude that a necessary and sufficient condition for the stability of the Z-pinch when \( m \neq 0 \) is

\[
2 \mu_0 r \frac{dp_0}{dr} + m^2 B_\theta^2 > 0
\] (30.23)

at all points in the fluid.
We can use the equilibrium condition, Equation (30.15), to eliminate the pressure gradient from the Equation (30.23). The result is

$$2B_0 \frac{d}{dr}(rB_0) < m^2 B_0^2.$$  \hspace{1cm} (30.24)

This can be re-written in either of two forms. The first is

$$\frac{r^2}{B_0} \frac{d}{dr}\left(\frac{B_0}{r}\right) < \frac{1}{2}(m^2 - 4).$$  \hspace{1cm} (30.25)

For , $B_0 \sim r$, so $B_0/r$ is constant. In this limit, the stability condition is therefore $m^2 > 4$, so that the interior of the Z-pinch is stable for stable for $|m| > 2$ and marginal for $|m| = 2$. For $r \to \infty$, $B_0 \sim 1/r$ and $d(B_0/r)/dr \sim 1/r^3 \to 0$, so the same conclusion holds in this limit.

The second form of the stability condition is

$$\frac{1}{B_0^2} \frac{d}{dr}(rB_0^2) < m^2 - 1.$$  \hspace{1cm} (30.26)

For $r \to \infty$, $d(rB_0^2)/dr \sim -1/r^2 \to 0$, and the stability condition is $m^2 > 1$. Therefore, all $|m| > 1$ are stable, and $|m| = 1$ is marginal, in this limit. For $r \to 0$, $rB_0^2 \sim r^3$ and the left hand side of Equation (30.26) $\sim r > 0$, so that the core is always unstable to the $|m| = 1$ mode.

Note that, since $J_\parallel = 0$, this $|m| = 1$ mode is not a current driven mode. Rather, stability is determined by a competition between field line bending (stabilizing) and unfavorable curvature (destabilizing). The latter wins out in the core of the Z-pinch for the mode with $|m| = 1$.

We now consider the case $m = 0$. Here $\mathbf{B} \cdot \nabla = 0$ everywhere, and so $\nabla \cdot \xi \neq 0$. After a formidable calculation, $\delta W$ is found to be

$$\frac{\delta W}{L} = \frac{\pi}{\mu_0} \int_0^a dr \left\{ \frac{r \Gamma p_0}{\Gamma p_0 + B_0^2 / \mu_0} \left( \frac{r \xi}{r} \right)' + 2 r p_0' \right\} \frac{\xi^2}{r^2},$$  \hspace{1cm} (30.27)

with

$$\xi_\zeta = \frac{i}{\Gamma p_0 + B_0^2 / \mu_0} \left[ \frac{r B_0^2}{\mu_0} \left( \frac{\xi}{r} \right)' + \Gamma p_0 \frac{r}{r} \left( r \xi_\zeta \right)' \right]$$  \hspace{1cm} (30.28)

for the minimizing perturbation. Again, stability requires that the integrand of Equation (30.27) be positive for all $r$, which leads to the stability condition for $m = 0$ modes:

$$-r \frac{d p_0}{p_0} = - \frac{d \ln p_0}{d \ln r} < \frac{4 \Gamma}{2 + \Gamma \beta_0},$$  \hspace{1cm} (30.29)
where the poloidal beta is $\beta_0 = 2\mu_0 p_0 / B_0^2$. The Z-pinch can support a pressure gradient as long as it is not too large.

We remark that, since the condition (30.29) depends on the adiabatic index $\Gamma$, it requires that the fluid satisfy the adiabatic law. This is seldom the case for real plasmas. In that case, all bets are off as far as $m = 0$ stability is concerned.

**The General Screw Pinch**

You may have observed that the stability calculations become increasingly formidable as the equilibrium becomes more complex. In this regard, the case of the General Screw Pinch inherits some of the worst elements of the calculations for the $\theta$-pinch and the Z-pinch. It will therefore be treated here with even more informality.

For the General Screw Pinch, $\mathbf{B} = B_\theta(r)\hat{e}_\theta + B_z(r)\hat{e}_z$, and the equilibrium condition is

$$
\frac{d}{dr} \left( p_0 + \frac{B^2_\theta + B^2_z}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0 .
$$

(30.30)

The minimizing perturbation has

$$
\mathbf{B} \cdot \nabla \left( \frac{\xi}{B} \right) = -\nabla \cdot \xi_\perp .
$$

(30.31)

In this case we can write $\mathbf{B} \cdot \nabla = i F(r)$, where $F(r) \equiv \mathbf{k} \cdot \mathbf{B} = m B_\theta / r + k B_z$. Therefore, $\mathbf{B} \cdot \nabla$ is well behaved everywhere that $F \neq 0$. In that case the minimizing perturbation has

$$
\xi_\parallel = \frac{i B}{F} \nabla \cdot \xi_\perp .
$$

(30.32)

The roots $r_0$ of the equation $F = 0$ are called singular surfaces (again associating the radial coordinate with flux surfaces), for at these points $\mathbf{B} \cdot \nabla$ is singular and cannot be inverted. From our discussion in Section 29, we can still choose a “well behaved” $\xi_\parallel$ that still minimizes $\delta W$, so that we can draw reliable conclusions from the Energy Principle. Physically, the surfaces $r_0$ are associated with $k_\parallel = 0$, so that the field line bending term is minimized. We may expect these surfaces to play an important role in determining stability.

Under these circumstances, and “after some algebra”, one finds

$$
\frac{\delta W}{L} = \frac{\pi}{\mu_0} \int_0^a \left( f |\xi'|^2 + g |\xi|^2 \right) dr ,
$$

(30.33)

where

$$
f = \frac{r F^2}{k_0^2} ,
$$

(30.34)

and
\[
g = 2 \frac{k^2}{k_0^2} (\mu_0 p'_0) + \left( \frac{k_0^2 r^2 - 1}{k_0^2 r^2} \right) r F^2 + \frac{2 k^2}{r k_0^4} \left( k B_z - \frac{m B_0}{r} \right) F .
\]

Some general remarks can be made.

1. First, \( f \geq 0 \) for all \( r \), so the term \( f|\bar{\xi}_r|^2 \) is stabilizing. However, it vanishes at the singular surfaces \( r_0 \) where \( k \cdot B = 0 \), so that we may expect instability to be associated with these radii.

2. The sign of the term \( g|\bar{\xi}_r|^2 \) is determined by the sign of \( g \). The minimizing displacement therefore should have \( |\bar{\xi}_r|^2 > 0 \) where \( g < 0 \), and \( |\bar{\xi}_r|^2 = 0 \) where \( g > 0 \).

It turns out that the analysis of pressure driven modes arising from the \( p'_0 \) term requires a detailed analysis of the behavior of the solutions of the Euler equation (the ideal MHD wave equation) near its “regular singular points”. This will be briefly outlined in Section 31.

For current driven modes (\( p'_0 = 0 \)), the sign of \( g \) is determined primarily (but not completely) by the sign of \( F \). For the case of a “straight torus” when \( k \) is quantized, we can write

\[
F = B_0 \left( m + nq \right) ,
\]

where

\[
q(r) = \frac{r B_z(r)}{R B_0(r)}
\]

is called the safety factor, for reasons that will become clear shortly. Consider an unwrapped “flux surface”, i.e., a cylinder of radius \( r \) unwrapped and layed out flat. The magnetic field lines lie completely within such surfaces, as sketched in the figure.

The wrapping angle is \( \phi = \tan^{-1} B_0 / B_z \). We define the pitch of the field lines in the surface as \( P(r) = r B_z / B_0 \). This is the distance that one would travel axially (in the \( z \)-
direction) by following a field line through one circuit from \( \theta = 0 \) to \( \theta = 2\pi \). The pitch is a function of radius, meaning that the wrapping angle varies varies from surface to surface. The safety factor is therefore the normalized pitch, \( q = P / R \). The quantity \( rq' / q \equiv d \ln q / d \ln r \) is called the magnetic shear.

From Equation (30.36) we see that when \( F = 0 \), \( q = -m / n \), i.e., \( q \) is a rational number. This is why these surfaces are also called rational surfaces. The field lines close upon themselves after \( m \) turns in the poloidal (\( \theta \)) direction and \( n \) turns in the toroidal (axial, or \( z \)) direction, within the surface.

The safety factor at the outer boundary \( r = a \) is \( q(a) = aB_z(a) / RB_\theta(a) \). Since \( B_\theta(a) \sim I / a \), where \( I \) is the total toroidal (axial) current, \( q(a) \sim a^2B_z(a) / I \sim \) (total toroidal flux)/(total toroidal current). The more current for a given flux, the smaller \( q(a) \).

The configuration is unstable if \( g < 0 \), which can only occur if \( F < 0 \), or \( q(r) < -m / \) \( n \). (In the tokamak literature it is customary to write \( n \rightarrow -n \), so that \( q < m / n \). Since it is only the relative orientation of \( k_\theta = m / r \) and \( k_z = k = n / R \) that enter the theory, it is also customary to consider only \( m \geq 0 \). Here we employ the standard mathematical notation for the Fourier decomposition and live with the minus sign.) Therefore, restricting the discussion to \( m > 0 \), the configuration is stable if \( n > 0 \), and may be unstable if \( n < 0 \) and \( q < m / \) \( |n| \).

The \( m = 1 \) mode may be unstable if \( q(r) < 1 / \) \( |n| \). We therefore must require \( q(r) \) > 1 everywhere to assure stability. In particular, at \( r = a \) we require \( B_z(a) / B_\theta(a) > R / a \). This is the Kruskal-Shafranov stability condition. It means that the ratio of the toroidal (axial) flux to the toroidal (axial) current cannot be too small. It implies the necessity of a strong toroidal magnetic field for stability. This field, which does not contribute to confinement, must be supplied by external means. The Kruskal-Shafranov condition forms the basis for the design of the tokamak.

We have seen that the minimizing displacement must have \( |\xi| \) \( ^2 \) > 0 when \( g < 0 \), and \( |\xi| \) \( ^2 \) = 0 when \( g > 0 \). For the case when \( g \) is a monotonically increasing function of \( r \), and to the extent that the sign of \( g \) is determined by the sign of \( F \), the situation is sketched in the figure.
This is the general case for a tokamak. The “top hat” shape of the displacement is typical of the $m = 1$ mode in a tokamak. Note that the displacement vanishes outside the rational surface, so that it does not “feel” the wall. The most unstable $m = 1$ displacement in a tokamak will consist (approximately) of a rigid shift of the flux surfaces inside the rational surface, and no displacement outside the rational surface. We emphasize that this discussion, and the figure, are heuristic and approximate because of the stabilizing term in $g$ proportional to $F^2$; the root of $g = 0$ does not exactly correspond to the root of $F = 0$. However, the picture serves as a reasonable guide to what happens when a more detailed analysis is attempted.

If instead $g$ is a monotonically decreasing function of $r$, then the situation is like that shown in the figure.

The minimizing displacement now must vanish inside the rational surface, and be non-zero outside of it. However, it must also satisfy the boundary condition $\xi_\alpha(a) = 0$. Therefore $\xi' \neq 0$, and the first term in Equation (30.34) can contribute to stabilization. The details depend on the relative location of the wall and the rational surface. This type of stabilization is called wall stabilization. A monotonically decreasing profile is characteristic of a Reversed-field Pinch (or RFP). We will describe this concept in more detail when we discuss plasma relaxation in a later Section.

This concludes our discussion of the Energy Principle. It has provided a means of deducing some very general properties of MHD stability, and we never had to solve a differential equation! This will not be the case from now on. As mentioned, the properties of the Euler equation near a regular singular point must be investigated in order to determine the stability of pressure driven modes. And, the Energy Principle is no longer valid in the presence of resistivity, so of necessity we must address differential equations when we study resistive instabilities. That is what follows.

Of course, all of the calculations of this Section can be (and have been) carried out in toroidal geometry, but the primary concepts remain unchanged.