37. MHD RELAXATION: MAGNETIC SELF-ORGANIZATION

Magnetized fluids and plasmas are observed to exist naturally in states that are relatively independent of their initial conditions, or the way in which the system was prepared. Their properties are completely determined by boundary conditions and a few global parameters, such as magnetic flux, current, and applied voltage. Successive experiments carried out with the same global parameters yield the same state, even though they were not initiated in exactly the same way (for example, how the gas initially fills the vacuum chamber, or the breakdown process). Further, if the system is disturbed it tends to return to the same state. These preferred states are called relaxed, or self-organized, states, and the dynamical process of achieving these states is called plasma relaxation, or self-organization. Relaxed states cannot result from force balance or stability considerations alone, because there may be many different stable equilibria corresponding to a given set of parameters and boundary conditions. Some other process must be at work.

The energy principle says that a system tries to achieve its state of minimum potential energy

\[ W = \int_{V_0} \left( \frac{B^2}{2\mu_0} + \frac{p}{\Gamma - 1} \right) dV. \]  

(37.1)

Taken literally, minimization of \( W \) yields the state \( B = 0, \ p = 0 \), which is physically irrelevant. Clearly, the minimization must be constrained in some way. Minimization with the condition that the total magnetic flux \( \Phi \) be fixed yields \( B = \text{constant}, \ p = 0 \), which is better but still not physically realistic. Further constraints are required.

Recall that, in ideal MHD, the integrals

\[ K_l = \int_{V_l} A \cdot B dV, \quad l = 1, 2, ..., \]  

(37.2)

are constant on each and every flux tube \( V_l \) in the system. These are the Wöltjer invariants. There are an infinite number of these constraints. In an earlier Section we showed that the existence of the Wöltjer invariants is equivalent to the assumption of ideal MHD, \( E = -\nabla \times B \), and vice versa. We therefore seek to minimize \( W \) subject to the constraint of ideal MHD. If we vary the magnetic field and pressure independently, then

\[ \delta W = 0 = \int_{V_0} \left( \frac{B \cdot \delta B}{2\mu_0} + \frac{\delta p}{\Gamma - 1} \right) dV, \]

\[ = \frac{1}{\mu_0} \int_{V_0} B \cdot \nabla \times (\xi \times B) dV + \frac{1}{\Gamma - 1} \int_{V_0} \delta p dV, \]  

(37.3)

where \( \xi \) is the displacement. Since \( \delta B \) and \( \delta p \) are independent, the last integral is minimized by setting \( \delta p = 0 \). The first integrand is rewritten using the vector identities
\[ \nabla \cdot \left[ B \times (\xi \times B) \right] = B \cdot \nabla \times (\xi \times B) - (\xi \times B) \cdot \nabla \times B , \]  
(37.4)

and
\[ B \times (\xi \times B) = \xi B^2 - B \xi \cdot B , \]  
(37.5)

so that
\[ \int_{V_0} (\xi \times B) \cdot \nabla \times B \, dV + \oint_{S} \hat{n} \cdot \left[ \xi B^2 - B \xi \cdot B \right] \, dS = 0 \]  
(37.6)

The surface integral vanishes with the boundary conditions \( \hat{n} \cdot \xi = \hat{n} \cdot B = 0 \), and the volume integral becomes
\[ \int_{V} \xi \cdot \left[ (\nabla \times B) \times B \right] \, dV = 0 \]  
(37.7)

Since this must hold for arbitrary \( \xi \), minimization of \( W \) requires
\[ (\nabla \times B) \times B = 0 \]  
(37.8)

or
\[ \nabla \times B = \lambda(r)B , \]  
(37.9)

where \( \lambda(r) = J \cdot B / B^2 \) is related to the parallel current density. It is a function of space that satisfies the equation
\[ B \cdot \nabla \lambda = 0 \]  
(37.10)

It is constant along field lines. The relaxed magnetic fields are force-free.

We remark on the minimization with respect to the pressure. Instead of setting \( \delta p = 0 \), we could have used the adiabatic (ideal MHD) energy equation
\[ \delta p = \Gamma \rho \nabla \cdot \xi - \xi \cdot \nabla \rho \]  
(37.11)

Minimization then yields \( \nabla p = (\nabla \times B) \times B \), instead of Equation (37.8). The pressure will be constant along a flux tube, but can vary from flux tube to flux tube. In that case, the pressure distribution would be determined by the details of the way the system was prepared, and would be unrepeatable. This is not what is observed. Instead, if there is a small amount of resistivity the flux tubes will break. The pressure will mix and equilibrate, resulting in a state with \( \nabla p = 0 \). To quote Taylor (J. B. Taylor, Rev. Mod. Phys. 58, 741(1986)): “Relaxation proceeds by reconnection of lines of force, and during this reconnection plasma pressure can equalize itself so that the fully relaxed state is a state of uniform pressure. Hence, the inclusion of plasma pressure does not change our conclusion about the relaxed state. Of course, one may argue that pressure relaxation might be slower than field relaxation, so that the former was incomplete and some residual pressure gradients would remain..... However, no convincing argument for determining the residual pressure gradient has yet been given. We shall, therefore, consider \( \nabla p \) to be negligible in relaxed states – which in any event is a good approximation for low-\( \beta \) plasmas.”
Dependence on the initial conditions is also a problem for the force-free relaxed states given by Equations (37.9) and (37.10). The function $\lambda(r)$ is determined by the way the system is prepared, which is uncontrollable. We conclude that ideal MHD over-constrains the system.

Taylor recognized that in a slightly resistive plasma contained within a perfectly conducting boundary, one flux tube will retain its integrity, and that is the flux tube containing the entire plasma! Then only the single quantity

$$K_0 = \int_{V_0} A \cdot B dV \quad (37.12)$$

will remain invariant. We recognize this as the total magnetic helicity. Note that this is not a proof; rather it is a conjecture based upon physical insight. Taylor's conjecture is then that MHD systems tend to minimize their magnetic energy subject to the constraint that the total magnetic helicity remains constant. In order to carry out this calculation we need to know something about how constrained minimization is expressed in the calculus of variations.

**Constrained Variation and Lagrange Multipliers**

Problem I: Given a continuous function $f(x,y,...)$ of $N$ variables in a close region $G$, find the point $(x_0,y_0,...)$ where $f$ has an extremum.

Solution I: Set

$$\frac{\partial f}{\partial x} = 0 \ ,$$

$$\frac{\partial f}{\partial y} = 0 \ ,$$

etc.

This yields $N$ simultaneous equations in $N$ unknowns whose solution is $x = x_0$, $y = y_0$, etc.

Problem II: Now suppose that the variables $(x,y,...)$ are no longer independent, but are subject to the restrictions, or constraints,

$$g_1(x,y,...) = 0 \ ,$$

$$g_2(x,y,...) = 0 \ ,$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$g_h(x,y,...) = 0 \ ,$$

where $h < N$. Find $(x_0,y_0,...)$.

Solution IIA: Use Equations (37.14) to algebraically eliminate $h$ of the unknowns. Then the procedure of Solution I yields $N - h$ simultaneous equations in $N - h$ which can be solved for $(x_0,y_0,...)$. This can be quite tedious.
Solution IIB: Introduce \( h + 1 \) new parameters \( \lambda_0, \lambda_1, ..., \lambda_h \), and construct the function
\[
F = \lambda_0 f + \lambda_1 g_1 + \lambda_2 g_2 + ... + \lambda_h g_h .
\] (37.15)
The unknowns are now \((x, y, ..., \lambda_0, \lambda_1, ..., \lambda_h)\). There is one more unknown than equations, so we can determine \((x_0, y_0, ...)\) and the ratios of \((\lambda_0, \lambda_1, ...)\) from the unconstrained problem
\[
\frac{\partial F}{\partial x} = 0 , \quad \frac{\partial F}{\partial y} = 0 , \quad ...... \quad (37.16)
\]
If \( \lambda_0 \neq 0 \), we can set \( \lambda_0 = 1 \) since \( F \) is homogeneous in the \( \lambda_i \). This procedure avoids the algebra of eliminating the unknowns from the constraints. The \( \lambda_i \) are called Lagrange multipliers, and the procedure is called the Method of Lagrange Multipliers.

We now apply this method to the constrained variational problem.

Problem III: Find \( y(x) \) that makes
\[
J \{ y \} = \int_{x_0}^{x_f} F(x, y, y')dx
\] (37.17)
stationary, has given boundary values \( y(x_0) = y_0 \), \( y(x_f) = y_1 \), and is subject to the subsidiary condition (constraint)
\[
K = \int_{x_0}^{x_f} G(x, y, y')dx = C .
\] (37.18)

Solution III: Let \( y(x) \) be the desired extremal, and consider the neighboring curve
\[
y + \delta y = y(x) + \varepsilon_1 \eta(x) + \varepsilon_2 \zeta(x) \quad ,
\] (37.19)
with \( \eta(x_0) = \eta(x_f) = \zeta(x_0) = \zeta(x_f) = 0 \). Then
\[
\Phi(\varepsilon_1, \varepsilon_2) = \int_{x_0}^{x_f} F(x, y + \varepsilon_1 \eta + \varepsilon_2 \zeta, y' + \varepsilon_1 \eta' + \varepsilon_2 \zeta')dx
\] (37.20)
must be stationary at \( \varepsilon_1 = \varepsilon_2 = 0 \) with respect to all sufficiently small values of \( \varepsilon_1 \) and \( \varepsilon_2 \) for which
\[
\Psi(\varepsilon_1, \varepsilon_2) = \int_{x_0}^{x_f} G(x, y + \varepsilon_1 \eta + \varepsilon_2 \zeta, y' + \varepsilon_1 \eta' + \varepsilon_2 \zeta')dx = C .
\] (37.21)

Let
\[
\chi = \lambda_0 \Phi(\varepsilon_1, \varepsilon_2) + \lambda \Psi(\varepsilon_1, \varepsilon_2) \quad ,
\] (37.22)
where \( \lambda_0 \) and \( \lambda \) are Lagrange multipliers. Then for an extremum, we require
\[
\frac{\partial \lambda}{\partial \varepsilon_1}\bigg|_{\varepsilon_1=0, \varepsilon_2=0} = \frac{\partial}{\partial \varepsilon_1} \left[ \lambda_0 \Phi + \lambda \Psi \right] = 0 ,
\]
(37.23)

and
\[
\frac{\partial \lambda}{\partial \varepsilon_2}\bigg|_{\varepsilon_1=0, \varepsilon_2=0} = \frac{\partial}{\partial \varepsilon_2} \left[ \lambda_0 \Phi + \lambda \Psi \right] = 0 .
\]
(37.24)

Using the results from our previous Section on the Calculus of Variations, we have
\[
\frac{\partial \Phi}{\partial \varepsilon_1} = \int_{x_0}^{x_1} \eta [F]_y \, dx ,
\]
(37.25)
\[
\frac{\partial \Phi}{\partial \varepsilon_2} = \int_{x_0}^{x_1} \zeta [F]_y \, dx ,
\]
(37.26)
\[
\frac{\partial \Psi}{\partial \varepsilon_1} = \int_{x_0}^{x_1} \eta [G]_y \, dx ,
\]
(37.27)
and
\[
\frac{\partial \Psi}{\partial \varepsilon_2} = \int_{x_0}^{x_1} \zeta [G]_y \, dx ,
\]
(37.28)

where we have introduced the notation
\[
[F]_y \equiv \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) .
\]
(37.29)

Equations (37.23) and (37.24) then become
\[
\int_{x_0}^{x_1} \left\{ \lambda_0 [F]_y + \lambda [G]_y \right\} \eta \, dx = 0 ,
\]
(37.30)

and
\[
\int_{x_0}^{x_1} \left\{ \lambda_0 [F]_y + \lambda [G]_y \right\} \zeta \, dx = 0 .
\]
(37.31)

From Equation (37.30), we find
\[
\frac{\lambda_0}{\lambda} = \frac{\int_{x_0}^{x_1} \eta [F]_y \, dx}{\int_{x_0}^{x_1} \eta [G]_y \, dx} ,
\]
(37.32)

so that the ratio \( \lambda_0 / \lambda \) is independent of \( \zeta \). Then since \( \zeta \) is arbitrary, we conclude from Equation (37.31) that
\[ \lambda_0 [F]_y + \lambda [G]_y = 0 \quad , \quad (37.33) \]

or \( \lambda_0 / \lambda = -[G]_y / [F]_y \). If \( \lambda_0 \neq 0 \) (i.e., \( [G]_y \neq 0 \)), we can set \( \lambda_0 = 1 \), and the minimizing condition is

\[ [F]_y + \lambda [G]_y = 0 \quad , \quad (37.34) \]

or

\[ \frac{d}{dx} \left( \frac{\partial}{\partial y} (F + \lambda G) - \frac{\partial}{\partial y} (F + \lambda G) \right) = 0 \quad . \quad (37.35) \]

So, minimizing \( \int F dx \) subject to the constraint \( \int G dx = C \) is equivalent to minimizing \( \int (F + \lambda G) dx \) without constraint.

Then, according to Taylor’s conjecture, we should minimize the functional \( I = W - \lambda' K_0 \) without constraint (where \( \lambda' \) is a constant, and the minus sign is conventional; eventually \( \lambda' \) will be related to the variable \( \lambda \) used in Equations (37.9) and (37.10)), i.e., the proper variational problem is \( \delta I = \delta W - \lambda' \delta K_0 = 0 \). (Do not confuse \( \delta W \) and \( \delta K_0 \) with their use in the Energy Principle.)

Proceeding, we have

\[ \delta W = \frac{1}{\mu_0} \int \delta \mathbf{B} \cdot \mathbf{B} dV = \frac{1}{\mu_0} \int (\nabla \times \delta \mathbf{A}) \cdot \mathbf{B} dV \quad , \]

\[ = \frac{1}{\mu_0} \int \left[ \delta \mathbf{A} \cdot \nabla \times \mathbf{B} + \nabla \cdot (\delta \mathbf{A} \times \mathbf{B}) \right] dV \quad , \]

\[ = \frac{1}{\mu_0} \int \delta \mathbf{A} \cdot \nabla \times \mathbf{B} dV + \frac{1}{\mu_0} \int (\delta \mathbf{A} \times \mathbf{B}) \cdot \hat{n} dS \quad , \quad (37.36) \]

and

\[ \delta K_0 = \int_{V_0} \left( \delta \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \delta \mathbf{B} \right) dV \quad , \]

\[ = \int_{V_0} \left[ \delta \mathbf{A} \cdot \mathbf{B} + \nabla \cdot (\mathbf{A} \times \delta \mathbf{A}) + \delta \mathbf{A} \cdot \nabla \times \mathbf{A} \right] dV \quad , \]

\[ = 2 \int_{V_0} \delta \mathbf{A} \cdot \mathbf{B} dV + \int_{S_0} (\mathbf{A} \times \delta \mathbf{A}) \cdot \hat{n} dS \quad , \quad (37.37) \]

so that

\[ \delta I = \int_{V_0} \delta \mathbf{A} \cdot \left[ \frac{1}{\mu_0} \nabla \times \mathbf{B} - 2 \lambda' \mathbf{B} \right] dV + \int_{S_0} (\hat{n} \times \delta \mathbf{A}) \cdot (\mathbf{B} + \lambda' \mathbf{A}) dS \quad . \quad (37.38) \]

The surface \( S \) is a perfect conductor where we require \( \hat{n} \times \delta \mathbf{E} = -i \omega \hat{n} \times \delta \mathbf{A} = 0 \), so that the surface term vanishes. Then setting \( \delta I = 0 \),
\[
\int_{V_0} \delta A \cdot \left[ \frac{1}{\mu_0} \nabla \times B - 2 \lambda \mathbf{B} \right] dV = 0 . 
\]  
(37.39)

Since this must hold for arbitrary \( \delta A \), we obtain the minimizing condition as

\[
\nabla \times B = \lambda B ,
\]  
(37.40)

where \( \lambda \equiv 2 \mu_0 \lambda' \) is a constant. This means that the system has lost memory of the details of how it was prepared. States that satisfy Equation (37.40) are relaxed states. They are independent of the initial conditions, in agreement with experiment.

We will see that Taylor’s conjecture leads to states that agree with experimental results over a wide range of parameters. But why should it be true? Why should the helicity be invariant while the energy is minimized? For example, consider \( K_0 \). We showed in a previous section that

\[
\frac{dK_0}{dt} = -2 \int \mathbf{E} \cdot \mathbf{B} dV .
\]  
(37.41)

In ideal MHD, \( \mathbf{E} = -\nabla \times \mathbf{B} \) and \( dK_0 / dt = 0 \). However, in resistive MHD, \( \mathbf{E} = -\nabla \times \mathbf{B} + \eta \mathbf{J} \) and

\[
\frac{dK_0}{dt} = -2 \eta \int \mathbf{J} \cdot \mathbf{B} dV = O(\eta) \neq 0 ,
\]  
(37.42)

so that \( K_0 \) is not constant. Further,

\[
\frac{dW}{dt} = -\int \mathbf{J} \cdot \mathbf{E} dV = -\eta \int J^2 dV = O(\eta) \neq 0 ,
\]  
(37.43)

so that \( K_0 \) and \( W \) formally decay at the same rate! So, in what sense does \( W \) decay while \( K_0 \) remains constant?

What matters is the relative decay of energy with respect to helicity. The dynamical processes that are responsible for relaxation should dissipate \( W \) faster than \( K_0 \), even if they are at the same order in the resistivity. The ratio \( W / K_0 \) should be minimized.

Taylor envisioned relaxation to occur as a result of resistive MHD turbulence acting at small scales. If we measure time in units of the Alfvén time \( \tau_A \), then Equations (37.42) and (37.43) can be written non-dimensionally as

\[
\frac{dW}{dt} = -\frac{2}{S} \int J^2 dV ,
\]  
(37.44)

and

\[
\frac{dK_0}{dt} = -\frac{2}{S} \int \mathbf{J} \cdot \mathbf{B} dV ,
\]  
(37.45)

where \( S = \tau_B / \tau_A >> 1 \) is the Lundquist number. We write the magnetic field as \( \mathbf{B} = \sum \mathbf{B}_k e^{ik \cdot r} \rightarrow B_k e^{ik \cdot r} \) and the current as \( \mathbf{J} = \sum i \mathbf{k} \times \mathbf{B}_k e^{ik \cdot r} \rightarrow k B_k e^{ik \cdot r} \). Then at large \( k \),
\[
\frac{dW_k}{dt} \approx -\frac{2}{S} k^2 B_k^2 , \quad (37.46)
\]

and
\[
\frac{dK_0}{dt} \approx -\frac{2}{S} kB^2 . \quad (37.47)
\]

Now \(dW_k / dt = O(1)\) when \(k^2 B^2 / S \approx 1\), or \(k_w = BS^{1/2} \approx O(S^{1/2})\). This is the wave number at which \(W\) is dissipated. But at this wave number,
\[
\frac{dK_0}{dt} \bigg|_{k=k_w} \approx \frac{k_w B^2}{S} = B^3 S^{-1/2} = O(S^{-1/2}) \ll 1 . \quad (37.48)
\]

This suggests that small scale turbulence may dissipate energy more efficiently than helicity.

It can also be argued that \(K_0\) is preserved by long wavelength motions. To do this, we first need to define the helical flux. In cylindrical geometry, the condition is
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r B_r \right) + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0 . \quad (37.49)
\]

We define a new independent variable \(\phi = m \theta + n z / R\), where \(m\) and \(n\) are poloidal and toroidal mode numbers. Then Equation (37.49) becomes
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r B_r \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \left( m B_\theta + \frac{nr}{R} B_z \right) = 0 . \quad (37.50)
\]

This condition will be satisfied identically if
\[
r B_r = -\frac{\partial \chi_{m,n}}{\partial \phi} , \quad (37.51)
\]

and
\[
m B_\theta + \frac{nr}{R} B_z = \frac{\partial \chi_{m,n}}{\partial r} , \quad (37.52)
\]

where \(\chi_{m,n}\) is the helical flux function associated with mode numbers \((m,n)\). Integrating Equation (37.52) from 0 to \(r\), we find
\[
\chi_{m,n} = m \int_0^r B_\theta dr' + \frac{n}{R} \int_0^r B_z r' dr' ,
\]

\[
= m \psi + \frac{n}{R} \Phi , \quad (37.53)
\]

where \(\psi\) is the poloidal flux and \(\Phi\) is the toroidal flux.

Now consider the integral
\[
\hat{K} = \int F(r,t) A \cdot B dV . \quad (37.54)
\]
If \( F = 1 \), then \( \dot{K} = K_0 \). The rate of change of \( \dot{K} \) is

\[
\frac{d\dot{K}}{dt} = \int_{V_0} \left[ \frac{\partial F}{\partial t} A \cdot B + F \left( \frac{\partial A}{\partial t} \cdot B + F A \cdot \frac{\partial B}{\partial t} \right) \right] dV ,
\]

(37.55)

\[
= \int_{V_0} \left[ \frac{\partial F}{\partial t} A \cdot B - 2F E \cdot B + F \nabla \cdot (A \times E) \right] dV .
\]

(37.56)

The second term in the integrand vanishes in ideal MHD. The remainder can be written as

\[
\frac{d\dot{K}}{dt} = \int_{V_0} \left[ \frac{\partial F}{\partial t} A \cdot B - (A \times E) \cdot \nabla F + \nabla \cdot (FA \times E) \right] dV ,
\]

\[
= \int_{V_0} \left[ \frac{\partial F}{\partial t} A \cdot B - (A \times E) \cdot \nabla F \right] dV + \int_{S_0} F (\hat{n} \times E) \cdot AdS .
\]

(37.57)

The surface term vanishes because \( \hat{n} \times E = 0 \) on \( S_0 \), and in ideal MHD, \( A \times E = B (A \cdot V) - V (A \cdot B) \), so that, finally,

\[
\frac{d\dot{K}}{dt} = \int_{V_0} \left[ \left( \frac{\partial F}{\partial t} + V \cdot \nabla F \right) A \cdot B - (B \cdot \nabla F) A \cdot V \right] dV .
\]

(37.58)

Then, in ideal MHD, \( \frac{d\dot{K}}{dt} = 0 \) if a) \( \frac{dF}{dt} = \frac{\partial F}{\partial t} + V \cdot \nabla F = 0 \), so that \( F \) is co-moving with the fluid, and b) \( B \cdot \nabla F = 0 \), so that \( F \) is constant along field lines.

Both \( \psi \), the poloidal flux, and \( \Phi \), the toroidal flux, satisfy these conditions, as does any function \( F(\psi, \Phi) \). In particular, any function of the helical flux \( \chi_{m,n} \), defined in Equation (37.53), satisfies these conditions. Therefore, a mode with mode numbers \((m,n)\) preserves the invariants

\[
\dot{K}_\alpha (m,n) = \int_{V_0} \chi_{m,n}^\alpha A \cdot BdV , \quad \alpha = 0,1,2,.....
\]

(37.59)

However, in the realistic case where all modes are present, the only invariant preserved by all the modes is \( \dot{K}_0 = K_0 \), the global helicity invariant.

The above is a heuristic argument, as it relies on ideal MHD and resistivity is present. However, it gives more credence to the conjecture that minimizing \( I = W - \lambda K_0 \) is a plausible approach. It also suggests how relaxation may occur as a result of long wavelength motions, with low \((m,n)\), rather than by small scale turbulence. In any case, the real test is to compare the predictions of the theory with the results of experiment.

For the most part we will restrict ourselves to doubly-periodic cylindrical geometry. The curl of Equation (37.40) is

\[
\nabla^2 B + \lambda^2 B = 0 ,
\]

(37.60)

and the \( z \)-component is
\[ \nabla^2 B_z + \lambda^2 B_z = 0 \quad . \quad (37.61) \]

In cylindrical geometry, this becomes Bessel’s equation, with solutions of the form
\[ B_z = \sum_{m,k} a_{m,k} J_m (\alpha r) e^{i(m \theta + kz)} \quad , \quad (37.62) \]

with
\[ \alpha^2 = \lambda^2 - k^2 \quad . \quad (37.63) \]

Equations for the other components are similarly found. It can be shown (but not here!) that only 2 of these solutions can have minimum energy: azimuthally symmetric solutions with \( m = 0 \),
\[ \frac{B_z}{B_0} = J_0 (\lambda r) \quad , \quad (37.64) \]
\[ \frac{B_{\theta}}{B_0} = J_1 (\lambda r) \quad , \quad (37.65) \]

and
\[ \frac{B_r}{B_0} = 0 \quad ; \quad (37.66) \]

and, helical solutions with \( m = 0 \) and \( m = 1 \) components,
\[ \frac{B_z}{B_0} = J_0 (\lambda r) + a_{1,k} J_1 (\alpha r) \cos (\theta + kz) \quad , \quad (37.67) \]
\[ \frac{B_{\theta}}{B_0} = J_1 (\lambda r) + \frac{a_{1,k}}{\alpha} \left[ \lambda J_1' (\alpha r) + \frac{k}{\alpha r} J_1 (\alpha r) \right] \cos (\theta + kz) \quad , \quad (37.68) \]

and
\[ \frac{B_r}{B_0} = - \frac{a_{1,k}}{\alpha} \left[ k J_1' (\alpha r) + \frac{\lambda}{\alpha r} J_1 (\alpha r) \right] \sin (\theta + kz) \quad . \quad (37.69) \]

In all cases,
\[ B_0 = \frac{\lambda a}{2 J_1 (\lambda a)} \frac{\Phi}{2\pi} \quad , \quad (37.70) \]

where
\[ \Phi = 2\pi \int_0^a B_z r dr \quad (37.71) \]

is the total axial (or toroidal) flux. The helical distortions make no contribution to the toroidal flux, which is all carried by the azimuthally symmetric solution.

The azimuthally symmetric states, Equations (37.64) – (37.66), are called the Bessel Function Model, or BFM. They are shown in the figure below.
For these states, the total helicity and the toroidal flux are related to $\lambda a$ through
\[
\frac{K_0}{\Phi^2} = \frac{L}{2\pi a} \left\{ \frac{\lambda a \left[ J_0^2(\lambda a) + J_1^2(\lambda a) \right] - 2J_0(\lambda a)J_1(\lambda a)}{J_1^2(\lambda a)} \right\}.
\] (37.72)

The details of the relaxed state are therefore completely determined by the two invariants $K_0$ and $\Phi$: $K_0/\Phi^2$ determines $\lambda a$, and hence the field profiles, through Equation (37.72); then $\Phi$ and $\lambda a$ determine the field amplitude through Equation (37.70). The quantity $K_0/\Phi$ is related to the total volt-seconds available to sustain the discharge.

We now define two useful parameters:
\[
F \equiv \frac{B_z(a)}{B} = \frac{\pi a^2 B_z(a)}{\Phi},
\] (37.73)
where $\langle ... \rangle$ denotes the volume average, is called the field reversal parameter; and
\[
\Theta \equiv \frac{B_0(a)}{B} = \frac{\pi a I}{\mu_0 \Phi},
\] (37.74)
which is called the pinch parameter. The latter is related to the ratio of the total toroidal current to the total toroidal flux. For the BFM, it is easy to show that
\[
F = \frac{\lambda a J_0(\lambda a)}{2J_1(\lambda a)} ,
\] (37.75)
and
\[
\Theta = \frac{\lambda a}{2} ,
\] (37.76)
so that $F$ and $\Theta$ are related by
\[
F = \frac{\Theta J_0(2\Theta)}{J_1(2\Theta)} .
\] (37.77)
A plot of $F$ versus $\Theta$ for the azimuthially symmetric states is shown as the solid line in the figure below. This is an example of an $F-\Theta$ diagram.

The theory predicts that the toroidal field at the wall will reverse sign with respect to its value on axis when $\lambda a > 2.4$, or $\Theta > 1.2$.

Thus, setting $\Theta$ by adjusting the current and flux predetermines the shape of the magnetic field profiles and the value of the toroidal field at the outer boundary. The $F-\Theta$ defines a continuum of relaxed states, which could be “dialed in” by the operator of an experiment. Two regimes are of particular interest. The first corresponds to $\Theta << 1$. It is called the tokamak regime. In this case the fields are given by the small argument limits of the Bessel functions $J_0$ and $J_1$, so that $B_z \approx B_0$, $B_\theta = (B_0\Theta/2)(r/a)$, and $B_z/B_\theta = 1/\Theta >> 1$. These fields are sketched in the figure below.

The second regime corresponds to $\Theta > 1.2$, and is called the Reversed-field Pinch (or RFP) regime. The fields are as sketched as the solid lines in the figure following Equation (37.71).

For the helical states given by Equations (37.67) – (37.69), $\lambda a$ is now determined by the boundary condition $B_\theta(a) = 0$, i.e.,
\[ kJ'_i(\alpha a) + \frac{\lambda}{\alpha a} J_i(\alpha a) = 0 \quad , \]  
(37.78)

where \( \alpha = \sqrt{\lambda^2 - k^2} \) [see Equation (37.69)], and solutions exist only for discrete values of \( \lambda a \). There are no solutions when \( \lambda a < 3.11 \), or \( \Theta < 1.56 \). Only the azimuthally symmetric states exist for lower \( \Theta \). However, when \( \Theta > 1.56 \), the helical state has the lowest energy. The minimum energy corresponds to \( ka \approx 1.25 > 0 \). The amplitude of the helical distortion, \( a_{1,k} \), is then determined by \( K_0 / \Phi^2 \) [see Equation (37.72)] with \( \lambda a = 3.11 \).

The predictions of the theory can be summarized as follows:

1. As the volt-seconds (expressed as \( K_0 / \Phi^2 \)) increase, \( \Theta \) will increase.
2. As \( \Theta \) increases, \( B_z(a) \) will decrease.
3. For \( \Theta > 1.2 \), \( B_z(a) / B_z(0) < 0 \).
4. At \( \Theta = 1.56 \) there will be the onset of a helical distortion.
5. As \( K_0 / \Phi^2 \) is further increased, the amplitude of the helical distortion will increase, but \( \Theta \) will remain fixed at 1.56. The increase in volt-seconds does not drive more current; it is absorbed by the increased inductance due to the helical distortion of the plasma.

So, how does the relaxation theory compare with experiment? Very well in one case, pretty well in others, and not well in another.

**Multi-Pinch**

The multi-pinch is an axisymmetric toroidal plasma with a non-circular cross section; the poloidal cross section of the plasma exhibits equatorial, or up-down, symmetry, as shown in the figure below.

For such systems the periodic cylindrical approximation used in our previous discussions of this Section does not apply, and toroidal effects must be included.
The calculation of the relaxed states goes through in much the same way as given previously, except that Equation (37.60) is now expressed as a partial differential equation in the poloidal plane; the details will not be given here (see Taylor). It turns out that physically interesting axisymmetric \((n = 0)\) solutions can be found. The lowest energy state possesses up-down symmetry; in analogy with our previous discussion, this is the only solution for low values of \(K_0 / \Phi^2\). The field profiles are again parameterized by \(\lambda a\); \(K_0 / \Phi^2\) determines \(\lambda a\), and then either \(K_0\) or \(\Phi\) determines the amplitude. There are also solutions that are not up-down symmetric (but still axisymmetric). These are analogous to the helically distorted states in the cylinder. These solutions do not exist for \(\lambda a < 2.21\); when \(\lambda a = 2.21\) these states have the lowest energy. As the volt-seconds are increased from a low value, \(\lambda a\) increases until \(\lambda a = 2.21\). As more volt-seconds are applied, \(\lambda a\) (and hence the current) remains fixed while the amplitude of the up-down asymmetry increases.

These predictions are borne out well by experiment. The current saturation at \(\lambda a = 2.21\) and the increase in up-down asymmetry are all observed. Details such as the dependence of the saturation level on toroidal flux are also predicted by the theory with quantitative accuracy (see Taylor).

**Reversed-field Pinch**

For the case of the RFP there is qualitative agreement between theory and experiment. Toroidal field reversal is observed, but it occurs at a larger value of \(\Theta\) than predicted. [See the data points in the figure following Equation (37.77)]. The pressure is not zero, and the parallel current (i.e., \(\lambda a\)) is not constant throughout the plasma. However, most of the discrepancy occurs in the outer regions of the cylinder. These are sketched in the figures below.

Both profiles are nearly constant over the inner part of the discharge. The pressure corresponds to \(\beta \sim 0.1\). The value of \(\lambda\) in the core of the plasma is in good agreement with the value for the BFM [see Equation (37.76)]. Experimentally determined magnetic field profiles are shown as the points in the figure following Equation (37.71). Again, deviation from the predictions of the theory occur primarily near the outer boundary.

In the RFP, relaxation seems to be inhibited near the wall. This is because the fully relaxed condition \(\lambda = \text{constant}\) inconsistent with the boundary conditions for a resistive plasma at a perfectly conducting boundary. For a resistive plasma, the tangential electric field at the wall is
\[ \hat{n} \times E = (\hat{n} \cdot B) V - (\hat{n} \cdot V) B + \eta \hat{n} \times J. \]

(37.79)

Since \( \hat{n} \times E = \hat{n} \cdot B = \hat{n} \cdot V = 0 \) at a perfectly conducting boundary, we must also require \( \hat{n} \times J = 0 \), i.e., the tangential component of the current density must vanish. This is inconsistent with non-vanishing \( \lambda \); \( \lambda \) must decrease to zero at the wall. The RFP is said to exhibit incomplete relaxation. And, if the magnetic relaxation is necessarily incomplete, so is the pressure relaxation.

Further, the current saturation predicted to occur at \( \Theta = 1.56 \) is not observed. Recall that Taylor’s helical state has \( ka = 1.25 \). However, in a cylinder of finite period length (as would result from straightening a torus), the wave number \( k \) is quantized as \( k = n / R \) (with \( R \) the major radius). The predicted helical states thus require \( na / R = 1.25 \), or an aspect ratio \( R/a = n / 1.25 \), where \( n \) is an integer. If the experiment is not so constructed, the helical states are impossible. (Helical fluctuations with \( m = 1 \) are observed in RFP experiments, but they have \( ka < 0 \) so they are not related to the helical relaxed state. They do, however, play an important role in the dynamics or relaxation.) In contrast, in the multi-pinches the distinction is between states of different up-down, rather than toroidal (or axial), symmetry. Quantization is not required, and there is good agreement with experiment.

Similar comments apply to spheromak plasmas.

Tokamak

For the tokamak, relaxation theory predicts \( p = \text{constant}, \ B_z = \text{constant}, \) and \( B_\theta = \text{constant} \times r \) (i.e., \( J_z = \text{constant} \)). This is clearly not what is observed in experiments...except possibly after a “major disruption”, a sudden event in which confinement is lost and the current is quenched. That state is consistent with the predictions of Taylor’s theory (\( p = J = 0, \ B_z = \text{constant} \)). Perhaps a the major disruption is the manifestation of a “relaxation event” in the tokamak. But if so, why don’t they occur all the time? Why are tokamaks the leading candidate for a controlled fusion reactor?

This last discussion leads us to enquire into the dynamics responsible for relaxation. As mentioned, Taylor envisioned relaxation to result from small scale turbulence in a resistive plasma. If this were true, then it shouldn’t know about the global geometry in which it is acting; it shouldn’t care if it’s in a tokamak, an RFP, or a multi-pinches. It should apply equally well to all systems describable by resistive MHD.

However, we have seen that this is not what is found in experiments. There are large differences between relaxation (or lack thereof) in the multi-pinches, the RFP (and spheromak), and the tokamak. Perhaps the fundamental relaxation dynamics operates differently in these devices. Perhaps it knows about the geometry and the overall magnetic configuration. Recall that long wavelength motions can preferentially preserve the global helicity \( K_0 \) [see Equation (37.59) and the preceding discussion]. These modes occur differently in the tokamak and the RFP. This can be seen in their \( q \) (safety factor) profiles, as sketched below.
The RFP has a decreasing $q$-profile. There are many long wavelength \([low \ (m,n)]\) singular surfaces, and they become closely spaced near the location of the field reversal (the reversal surface). It is easy for them to interact non-linearly, and provide quasi-continuous relaxation. Large, quasi-periodic oscillations are common in RFP plasmas, as shown below.

They are called sawtooth oscillations (for historical reasons). They are characterized by (among other things) large increases in the amplitude of low \((m,n)\) magnetic fluctuations with helical pitch corresponding to resonant perturbations. It is well established both experimentally and theoretically as a result of numerical simulation that MHD relaxation is associated with these events.

The cyclic relaxation process that occurs in an RFP is indicated schematically in the diagram below.
In contrast, the tokamak has an increasing $q$-profile, and only a handful of low $(m,n)$ rational surfaces. These are widely separated so it is difficult for them to interact with each other. The magnetic configuration prevents relaxation from occurring, even though low level turbulence is always present. On the isolated occasion when these modes can seriously interact non-linearly, disruptive-like events are predicted. Perhaps this disruptive behavior is just the tokamak seeking its preferred state.