3. MASS CONSERVATION AND THE EQUATION OF CONTINUITY

We now begin the derivation of the equations governing the behavior of the fluid. We will start by looking at the mass flowing into and out of a physically infinitesimal volume element.

There are 2 “viewpoints”, and they are equivalent:

1. **Eulerian**: A volume element is fixed in space in the “laboratory” frame of reference.

2. **Lagrangian**: The surface of the volume element is co-moving with the fluid, in the “fluid” frame of reference.

We will use whichever is most convenient.

First consider the Eulerian picture. The volume element $dV$ is shown below:

$P(x_1, x_2, x_3)$ is the centroid of the volume element. The sides of the volume element are fixed in space. Fluid can flow into and out of the volume element through the sides.

Let the mass density at $P(x_1, x_2, x_3)$ be $\rho(x_1, x_2, x_3)$ (mass/volume). It is the average (and nearly uniform) mass density throughout $dV$. The total mass contained within $dV$ is

$$M = \int \rho dV = \int \rho dx_1 dx_2 dx_3 \ .$$  \hspace{1cm} (3.1)

Assume that there are no sources or sinks of mass within $dV$. Then $dM / dt = \text{the rate at which mass enters or leaves through the surface } dS$.

A surface element $dS$ is shown below.
The surface area is \(dS\), and \(\mathbf{n}\) is a unit vector normal (perpendicular) to the surface (in an average sense). When \(dS\) is a side of a volume element \(dV\), \(\mathbf{n}\) is assumed to point \textit{out} of the volume element (i.e., from inside to outside). The flux of mass (mass/unit area/unit time) passing through a surface is \(\rho \mathbf{V}\), where \(\mathbf{V}\) is the fluid velocity. It is a vector quantity (actually, a pseudovector, because of the presence of \(\rho\)). Then the mass per unit time flowing through \(dS\) is \(\rho \mathbf{V} \cdot dS = \rho \mathbf{V} \cdot \mathbf{n} dS\), and the total rate of flow of mass \textit{out} of the volume \(dV\) is

\[
\sum_{\text{faces}} \rho \mathbf{V} \cdot dS \Rightarrow \oint_{\mathcal{S}} \rho \mathbf{V} \cdot dS = \oint_{\mathcal{S}} \rho \mathbf{V} \cdot \mathbf{n} dS \, ,
\]

where the integral is over the surface enclosing \(dV\). Since this must be equal to \(-dM/dt\), we have

\[
\frac{dM}{dt} = \frac{d}{dt} \int_{\mathcal{V}} \rho dV = -\oint_{\mathcal{S}} \rho \mathbf{V} \cdot \mathbf{n} dS \, .
\]

For a fixed (Eulerian) surface, we can take the total time derivative inside the volume integral as a partial derivative:

\[
\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} dV = -\oint_{\mathcal{S}} \rho \mathbf{V} \cdot \mathbf{n} dS \, .
\]

By Gauss’ theorem,

\[
\oint_{\mathcal{S}} \rho \mathbf{V} \cdot \mathbf{n} dS = \int_{\mathcal{V}} \nabla \cdot (\rho \mathbf{V}) dV \, ,
\]

so that

\[
\int_{\mathcal{V}} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) \right] dV = 0 \, .
\]

This expression must hold for every arbitrarily shaped volume; the only way that it can be satisfied is if the integrand vanishes identically, or

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{V}) \, .
\]

This is called the \textit{continuity equation}. It expresses \textit{conservation of mass} in the Eulerian frame of reference.
We remark that Equation (3.7) is a partial differential equation with four dependent variables: $\rho$ and the three components of $\mathbf{V}$. If the velocity were known \textit{a priori}, the system would be closed and we could solve Equation (3.7) for the evolution of $\rho$. Problems in which the velocity field is fixed, or specified in advance, are called \textit{kinematic}. Problems where $\mathbf{V}$ is determined from other physical principles are called \textit{dynamic}, and the latter is the case of interest here. We therefore have 3 more unknowns than we have equations; the problem is not \textit{closed}. This problem of \textit{closure} is of fundamental importance in MHD, and we will discuss it in more detail later in this course.

We now describe conservation of mass in the Lagrangian picture. Here the volume element $dV$ is co-moving with the fluid, as sketched in the figure below:

![Figure showing conservation of mass](image)

Every point on the surface and within the volume is moving with the local velocity $\mathbf{V} = dx / dt$; the coordinates of each “bit” of the volume element are thus time dependent: $x_i = x_i(t)$. The shape of the volume element can distort with time. However, since each point on the boundary $S$ moves with the fluid, no fluid can flow across the surface, so that \textit{the total mass within the volume element is fixed in time}: $dM / dt = 0$, and mass is automatically conserved.

However, things are still complicated. As the volume element moves through space, it’s total mass, as given by Equation (3.1), remains constant, but since the total volume of the element can change as it distorts due to fluid motions, the mass density must be considered to be a function of time. Conservation of mass is then stated as

$$\frac{dM}{dt} = 0 = \frac{d}{dt} \int_V \rho(t) dx_1(t) dx_2(t) dx_3(t) .$$

(3.8)

So, we not only need to calculate the change in $\rho$, but we need to account for the change in the volume $dV$ as it moves through space.
To this end we introduce a “new” infinitesimal, $\delta x_i$. The infinitesimal operator $\delta$ is taken to operate only on the spatial coordinates $x_i$; the notation is reserved to time. In all other respects, $d$ and $\delta$ are the same. Equation (3.8) is written as

$$\frac{d}{dt} \int \rho(t) \delta x_i(t) \delta x_2(t) \delta x_3(t) = 0 \quad \text{,}$$

(3.9)
and the time dependence of the coordinates is given by

$$\frac{dx_i}{dt} = V_i \quad \text{.}$$

(3.10)
Now time is the only independent variable. Differentiating under the integral sign in Equation (3.9), we have

$$0 = \int \left[ \frac{d\rho}{dt} \delta x_i \delta x_2 \delta x_3 + \rho \frac{d}{dt} \left( \delta x_i \delta x_2 \delta x_3 \right) \right] \quad \text{,}$$

$$= \int \left[ \frac{d\rho}{dt} \delta x_i \delta x_2 \delta x_3 + \rho \left( \frac{d\delta x_i}{dt} \delta x_2 \delta x_3 + \delta x_i \frac{d\delta x_2}{dt} \delta x_3 + \delta x_i \delta x_2 \frac{d\delta x_3}{dt} \right) \right] \quad \text{,}$$

$$= \int \left[ \frac{d\rho}{dt} \delta x_i \delta x_2 \delta x_3 + \rho \delta x_i \delta x_2 \delta x_3 \left( \frac{\delta V_1}{\delta t} + \frac{\delta V_2}{\delta t} + \frac{\delta V_3}{\delta t} \right) \right] \quad \text{,}$$

(3.11)
where we have used the fact that $\delta$ and $d$ are both infinitesimals, along with Equation (3.10), to write $d(\delta x_i)/dt = \delta(dx_i/dt) = \delta V_i$. We recognize the last term in brackets as $\nabla \cdot \mathbf{V}$. Then writing $d\mathbf{V} = \delta x_i \delta x_2 \delta x_3$,

$$0 = \int \left[ \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{V} \right] d\mathbf{V} \quad \text{.}$$

(3.12)
As with Equation (3.6), since this must hold for arbitrary volume elements we require

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{V} = 0 \quad \text{.}$$

(3.13)
This is the expression for conservation of mass in the Lagrangian frame of reference.

Equation (3.13), the Lagrangian expression, appears to be different from Equation (3.7), the Eulerian expression. In particular, how are we to interpret the total time derivative $d/\!\!d t$ that appears in Equation (3.13)? Since these equations each express the law of conservation of mass, they must be consistent. Note that we can write Equation (3.7) as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} = 0 \quad \text{,}$$

(3.14)
which will be consistent with Equation (3.13) if we identify

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho \quad \text{.}$$

(3.15)
Generally, the operator \( \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \) is called the total time derivative, or the Lagrangian derivative. It measures the total change in a quantity associated with a fluid element as it moves about in space. This can be seen as follows. The Lagrangian change in the density in a time \( dt \) consists of two parts, \( d\rho = d\rho_1 + d\rho_2 \), where \( d\rho_1 \) is the change in \( \rho \) during \( dt \) at a fixed point in space,

\[
d\rho_1 = \frac{\partial \rho}{\partial t} dt , \tag{3.16}
\]

and \( d\rho_2 \) is the difference between densities separated by a distance \( dx \), at the same time \( t \),

\[
d\rho_2 = dx \cdot \nabla \rho . \tag{3.17}
\]

The total change in \( \rho \) is therefore

\[
d\rho = \frac{\partial \rho}{\partial t} dt + dx \cdot \nabla \rho , \tag{3.18}
\]

so that

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho . \tag{3.19}
\]

Of course, this result can also be obtained formally by applying the chain rule to \( \rho = \rho[x(t),t] \), i.e.,

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho ,
\]

but this provides little physical insight.

The term \( \mathbf{V} \cdot \nabla \rho \) is called the advective derivative. The terminology originated in weather and climate modeling, where convection refers to vertical uplift driven by buoyancy and thermal forces, and advection refers to wind driven horizontal transport. The terminology has carried over to MHD, where it refers to all velocity driven transport. Its origin is illustrated in the figure below:
The profile $\rho(x)$ is being carried to the right with velocity $V = dx/dt$; in a time $\Delta t$ it is carried a distance $V\Delta t$. The change in $\rho$ in a time $\Delta t$ at a fixed point $x_0$ is just $\Delta \rho = -(\partial \rho / \partial x) V \Delta t$, or $d\rho / dt = -V \cdot \nabla \rho$.

The term $-\rho \nabla \cdot \mathbf{V}$ measures the change in $\rho$ due to compression or dilation of the fluid element. This is illustrated in the figure below:

In the figure on the left, $\nabla \cdot \mathbf{V} > 0$, the flow is diverging, there is net flow out of the volume element, and the mass within the volume element is decreasing. In the figure on the right, $\nabla \cdot \mathbf{V} < 0$, the flow is converging, there is net flow into the volume element, and the mass within the volume element is increasing.

The continuity equation and conservation of mass are exactly the same in hydrodynamics and MHD.