5. ENERGY FLOW

The enumeration of the various sources and flows of energy is enabled by working in the Lagrangian frame of reference; i.e., we consider a volume element that is co-moving with the fluid. The heat input rate per unit volume is equal to the rate of energy flow across the surface plus the volumetric heating rate, i.e.,

\[
\rho \frac{dQ}{dt} = -\nabla \cdot \mathbf{q} + R_v ,
\]

where \( Q \) is the heat per unit mass, \( \mathbf{q} \) is the heat flux through the boundary, and

\[
R_v = \Pi : \nabla \mathbf{V} + \eta J^2 ,
\]

is the volumetric heating rate. The first term is the viscous heating rate, defined in Section 4, and the second term is the Ohmic heating rate.

We write the first law of thermodynamics (conservation of energy) as

\[
dQ = pd\left(\frac{1}{\rho}\right) + de ,
\]

where \( dQ \) is the change in heat per unit mass, \( pd(1 / \rho) \) is the PV work per unit mass, and \( de \) is the change in energy per unit mass. Substituting this into Equation (5.2), we have

\[
p\rho \frac{d}{dt}\left(\frac{1}{\rho}\right) + \rho \frac{de}{dt} = -\nabla \cdot \mathbf{q} + \Pi : \nabla \mathbf{V} + \eta J^2 .
\]

Now,

\[
\frac{d}{dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2} \frac{d\rho}{dt} = \frac{1}{\rho} \nabla \cdot \mathbf{V} ,
\]

where we have used the Lagrangian form of the continuity equation. Then Equation (5.4) becomes

\[
\rho \frac{de}{dt} = -p\nabla \cdot \mathbf{V} - \nabla \cdot \mathbf{q} + \Pi : \nabla \mathbf{V} + \eta J^2 .
\]

The term on the left hand side is the rate of change of energy per unit volume. It is equal to the sum of the work done to expand or compress the fluid element, the rate of heat flow through the surface, and the volumetric heating rate due to viscous and resistive processes. We can again use the continuity equation to write

\[
\rho \frac{de}{dt} = \frac{d}{dt} (\rho e) + \rho e \nabla \cdot \mathbf{V} ,
\]

or, in the Eulerian frame,

\[
\rho \frac{de}{dt} = \frac{\partial}{\partial t} (\rho e) + \nabla \cdot (\rho e \mathbf{V}) .
\]
Then the equation describing energy and heat flow in the Eulerian frame of reference is
\[
\frac{\partial}{\partial t} (\rho e) = -\nabla \cdot (\rho e \mathbf{V}) - p \nabla \cdot \mathbf{V} - \nabla \cdot \mathbf{q} + \Pi : \nabla \mathbf{V} + \eta J^2 .
\]  
(5.10)

Not surprisingly (and certainly not very originally), Equation (5.10) is called the energy equation.

To an excellent approximation, a plasma behaves as an ideal gas, i.e., the energy depends only on the pressure. This relationship is written as
\[
\rho e = \frac{p}{\Gamma - 1} ,
\]  
(5.11)

where \( \Gamma \) is the adiabatic index. For a plasma, \( \Gamma = 5/3 \). Using Equation (5.11), and the identity \( \nabla \cdot (p \nabla) + (\Gamma - 1) p \nabla \cdot \mathbf{V} = \Gamma p \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p \), we obtain a evolutionary equations for the pressure in both the Eulerian frame,
\[
\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p = -\Gamma p \nabla \cdot \mathbf{V} + (\Gamma - 1) \left[ -\nabla \cdot \mathbf{q} + \Pi : \nabla \mathbf{V} + \eta J^2 \right] ,
\]  
(5.12)

and the Lagrangian frame
\[
\frac{dp}{dt} = -\Gamma p \nabla \cdot \mathbf{V} + (\Gamma - 1) \left[ -\nabla \cdot \mathbf{q} + \Pi : \nabla \mathbf{V} + \eta J^2 \right] .
\]  
(5.13)

The first term on the right hand side of Equation (5.13) represents reversible PV work. The second term represents irreversible heating processes. If these are absent, we say that the fluid is ideal. In that case we can again use the continuity equation to eliminate \( \nabla \cdot \mathbf{V} \) and obtain
\[
\frac{d}{dt} \left( \frac{p}{\rho^\Gamma} \right) = 0 ,
\]  
(5.14)

so that as the fluid element moves about in space it obeys the so-called adiabatic law \( p / \rho^\Gamma = \text{constant} \).

We can then summarize the results of Sections 3, 4, and 5 by stating the fluid equations in Eulerian form:
\[
\frac{\partial p}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 ,
\]  
(5.15)

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \rho_\gamma \mathbf{E} - \nabla p + \mathbf{J} \times \mathbf{B} + \nabla \cdot \Pi ,
\]  
(5.16)

and
\[
\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p = -\Gamma p \nabla \cdot \mathbf{V} + (\Gamma - 1) \left[ -\nabla \cdot \mathbf{q} + \Pi : \nabla \mathbf{V} + \eta J^2 \right] .
\]  
(5.17)

Note that the advective derivative \( \mathbf{V} \cdot \nabla \) appears prominently in all the equations. It will also appear in the equations that describe the dynamics of the electromagnetic fields.
We again remark that Equations (5.15-17) are not closed, i.e., there are more unknowns than there are equations. In particular, we will have to say something about \( J \), \( B \), \( \Pi \), and \( q \). The first will come from electrodynamics. The second require a further discussion of closures. These are the next two topics.