6. THE ELECTROMAGNETIC FIELD

The electromagnetic fields are: \( E \), the electric field; and \( B \), the magnetic flux density, or magnetic field. The sources of these fields are the electric charge density, \( \rho_e \), and the electric current density, \( J \). Together, these must satisfy Maxwell’s equations:

**Faraday’s Law:**

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} ,
\]

(6.1)

**Ampère’s Law:**

\[
\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} ,
\]

(6.2)

**Gauss’ Law:**

\[
\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon_0} ,
\]

(6.3)

**Absence of magnetic monopoles:**

\[
\nabla \cdot \mathbf{B} = 0 .
\]

(6.4)

These equations are written in MKS units. This convention will be used throughout. In these units the square of the speed of light is

\[
c^2 = \frac{1}{\varepsilon_0 \mu_0} .
\]

(6.5)

The constant \( \varepsilon_0 \) is called the permittivity of free space, and the constant \( \mu_0 \) is called the permeability of free space.

The dynamics of the electromagnetic fields and the fluid are coupled through Ohm’s Law,

\[
\mathbf{E}' = \eta \mathbf{J} ,
\]

(6.6)

where \( \eta \) is the electrical resistivity, which is to be considered a material property of the fluid, and \( \mathbf{E}' \) is the electric field as seen by a conductor moving with velocity \( \mathbf{V} \). According to the general theory of relativity, this is given by

\[
\mathbf{E}' = \frac{\mathbf{E} + \mathbf{V} \times \mathbf{B}}{\sqrt{1 - \frac{V^2}{c^2}}} ,
\]

(6.7)

where \( \mathbf{E} \) is the electric field in the stationary frame.

Maxwell’s equations and Ohm’s Law are Lorentz invariant, i.e., they are physically accurate to all orders of \( V^2 / c^2 \). However, the fluid equations [Equations (5.15) – (5.17)] are Galilean invariant; they are physically accurate only to \( O(V / c) \). The two systems of equations are incompatible as presently formulated. So, we either need to make the fluid
equations relativistic (not a savory task!), or to render Maxwell’s equations Gallilean invariant. As stated in the Introduction (Section 1), in MHD we will consider only low frequencies, i.e., $V^2/c^2 = (\omega L/c)^2 << 1$. We therefore choose the latter course, and seek a form of Maxwell’s equations that is only accurate through $O(V/c)$.

Consider Ohm’s Law, Equation (6.6). From Equation (6.7), when $V^2/c^2 << 1$ we can write the electric field in the moving frame as

$$E' = (E + \mathbf{V} \times \mathbf{B}) \left(1 - \frac{1}{2} \frac{V^2}{c^2} + \ldots \right) ,$$

$$E + \mathbf{V} \times \mathbf{B} + O\left(\frac{V^2}{c^2}\right) . \quad (6.8)$$

Ohm’s Law then becomes

$$E + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J} \ , \quad (6.9)$$

which is the proper MHD form. It is sometimes called the resistive Ohm’s Law. When $\eta = 0$, it is called the ideal MHD Ohm’s Law. Note that, for this ideal MHD, the electric field scales like $E_0 \sim V_0 B_0$, or $V_0 \sim E_0 / B_0$. We will find these useful in a moment.

Now consider Ampère’s Law, Equation (6.2). The ratio of the two terms on the right hand side is, approximately

$$\frac{1}{c^2} \frac{\partial E}{\partial t} \sim \frac{E_0 \omega / c^2}{B_0 / L} \sim \frac{V_0 \omega L}{c^2} \sim \frac{V_0^2}{c^2} << 1 \ , \quad (6.10)$$

where we have set $V_0 \sim \omega L$. We can therefore ignore the second term (the displacement current) compared with the first, and the low frequency version of Ampère’s Law is

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} \ . \quad (6.11)$$

In MHD, this equation defines the current density.

Next consider Gauss’ Law, Equation (6.3). When combined with the ideal MHD Ohm’s law, we have

$$\rho_q = -\varepsilon_0 \nabla \cdot (\mathbf{V} \times \mathbf{B}) \neq 0 \ , \quad (6.12)$$

so that MHD allows for a non-vanishing charge density. This net charge must arise from a difference $\Delta n$ between the local number densities of positive and negative charges. Then we can write $\Delta n / n_0 \sim E_0 \varepsilon_0 / (n_0 L e) \sim \varepsilon_0 V_0 B_0 / n_0 L e$, where $n_0$ is the average number density of positive and negative charges. Then using Equation (6.5) to eliminate $\varepsilon_0$, we find

$$\frac{\Delta n}{n_0} \sim \frac{V_0}{c^2} \frac{B_0^2}{\mu_0 n_0 M e B_0 L} \ , \quad (6.13)$$
where \( M \) is the mass of the individual positively charged particles (ions). We anticipate future results and identify \( V_A^2 = \Omega / (\mu_0 n_0 M) \) as the Alfvén speed. (This turns out to be the propagation speed of the shear waves briefly described in Section 1.) We also identify \( \Omega = eB_0 / M \) as the ion gyro-frequency (the frequency at which the individual ions orbit the magnetic field lines). Then setting \( V_0 \sim V_A \), we have

\[
\frac{\Delta n}{n_0} \sim \frac{V_0^2}{c^2} \frac{V_0}{\Omega L} .
\]

(6.14)

Finally, identifying \( V_A / \Omega L \equiv d_i / L \) (where \( d_i = c / \omega_{pi} \) is the ion skin depth and \( \omega_{pi} = n_0 e^2 / \epsilon_0 M \) is the square of the plasma frequency), we can estimate the size of the excess electric charge as

\[
\frac{\Delta n}{n_0} \sim \frac{d_i V_0^2}{L c^2} ,
\]

(6.15)

which is \(< V_0^2 / c^2 \) since \( d_i / L \ll 1 \). This result is called quasi-neutrality; it is a consequence of the low frequency assumption.

However, the charge density cannot be ignored if the parameters are such that \( \Delta n / n_0 \sim V_0 / c \). This can occur if \( (d_i / L)(V_0 / c) \sim 1 \), or on length scales \( L \sim (V_0 / c)d_i \). If we estimate \( V \sim V_{shi} \sim \sqrt{T / M} \), then \( L \sim \sqrt{\epsilon_0 T / n_0 e^2} = \lambda_D \), the Debye length. This is assumed to be much smaller than any macroscopic scale length.

The virtual vanishing of the electric charge density does not imply that the electrostatic field vanishes. In steady state (\( \partial / \partial t = 0 \)) Faraday’s Law requires \( \nabla \times E = 0 \), or \( E = -\nabla \phi \), and so the field is completely electrostatic, and can be large. Instead, regions of smooth field (where \( \nabla \cdot E \sim 0 \)) are “patched together” across layers with finite charge density and thickness that is vanishingly small, i.e., \( O(\lambda_D) \). This is reminiscent of (although not completely analogous to) the role of shock waves in hydrodynamics.

Finally, it can be shown that the ratio of the electric force to the Lorentz force is

\[
\left| \frac{\rho_e E}{|J \times B|} \right| \sim \frac{V^2}{c^2} \ll 1 ,
\]

(6.16)

so that it can be dropped from the equation of motion. The charge density therefore never enters the MHD equations. However, if you ever want to know what it is, all you have to do is compute \( \rho_q = -\epsilon_0 \nabla \cdot (\nabla \times B) \) (at least in ideal MHD).

In Eulerian form, the final equations of the MHD model are:

Equations for the fluid:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 ,
\]

(6.17)
\[
\begin{align*}
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) &= -\nabla p + \mathbf{J} \times \mathbf{B} + \nabla \cdot \mathbf{\Pi}, \\
\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p &= -\Gamma p \nabla \cdot \mathbf{V} + (\Gamma - 1) \left[ -\nabla \cdot \mathbf{q} + \mathbf{\Pi} : \nabla \mathbf{V} + \eta J^2 \right],
\end{align*}
\] (6.18) (6.19)

Equations for the electromagnetic fields:
\[
\begin{align*}
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E},
\end{align*}
\] (6.20)
\[
\begin{align*}
\mu_0 \mathbf{J} &= \nabla \times \mathbf{B},
\end{align*}
\] (6.21)

Ohm’s Law, which couples the fluid and the fields:
\[
\begin{align*}
\mathbf{E} + \nabla \times \mathbf{B} &= \eta \mathbf{J}.
\end{align*}
\] (6.22)

Equations (6.20) and (6.21) are sometimes called the “pre-Maxwell equations”, because the represent the state of knowledge of the electromagnetic field before Maxwell’s introduction of the displacement current.

Equations (6.17) – (6.22) are 14 equations in 27 unknowns: \( \rho \) (1 unknown), \( \mathbf{V} \) (3), \( p \) (1), \( \mathbf{\Pi} \) (9), \( \mathbf{J} \) (3), \( \mathbf{B} \) (3), \( \mathbf{E} \) (3), \( \mathbf{q} \) (3), and \( \eta \) (1). The conditions \( \nabla \cdot \mathbf{B} = 0 \) implied by Faraday’s law, and \( \nabla \cdot \mathbf{J} = 0 \) implied by Ampère’s Law, either increases the number of equations by 2 or decreases the number of unknowns by 2, depending on your point of view. So, we need expressions for 13 of the variables in terms of the other 14. This is the problem of closure. It will be discussed next.