Range of Validity of Estimate of Spring-mass Correction in Uniform Circular Motion

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In Lab M6 for Physics 201/207, a rotating turntable is used to provide azimuthal forces on a mass-spring system. The mass seeks radial equilibrium under the action of centrifugal force and the force due to the spring. In this note we explore the range of validity of a simple estimate of the correction due to non-zero mass of the spring, by comparison with an exact calculation (cf. R. Weinstock, *Spring-mass Correction in Uniform Circular Motion*, AJP, **32**, 370 (1964)).

I. INTRODUCTION

Consider a massless spring with unstretched length L_0 that is linear, i.e. if it is stretched to length L, each point in the spring is under a tension $T = -k(L - L_0)$ and the spring as a whole has potential energy $U = \frac{1}{2}k(L - L_0)^2$. Attach one end of the spring to the axis of a rotating table (axis of rotation being parallel to local gravity), and attach a mass m to the other end. Rotate the table with angular frequency ω, enforcing that the table provides all azimuthal forces necessary to keep the spring and mass aligned along a radius at all times. The mass m will find equilibrium when the spring is stretched to a length L such that

$$k(L - L_0) = m\omega^2 L \ .$$

What will happen if the mass m_0 of the spring differs from 0?

II. ESTIMATE

A diagram of the situation is shown in Fig. 1. Assume the spring stretches uniformly under the centrifugal force due to its mass and the mass m. Under these conditions, the kinetic energy of the block is zero (in the rotating frame) and the equilibrium condition reads

$$\frac{dE}{dL} = \frac{dV}{dL} = \frac{d}{dL}\left[\frac{1}{2}k(L - L_0)^2 - \frac{1}{2}m\omega^2 L^2 - \int_0^L \frac{dm}{dr} dr\omega^2 \frac{r^2}{2}\right] = 0,$$

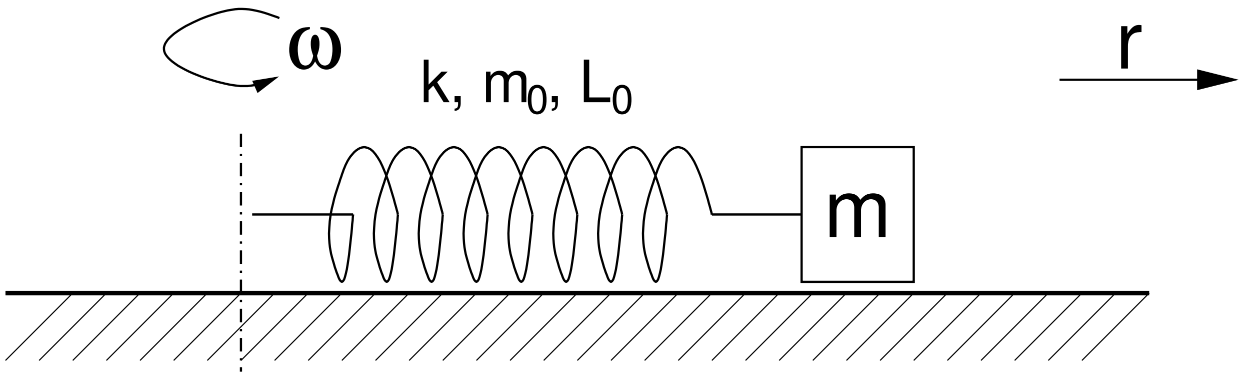

FIG. 1: Diagram of spring-mass set-up.

where L is both the radial position of the mass m and the length of the spring. This gives $k(L-L_0) = m\omega^2 L + \frac{d}{dL}\left[\frac{m_0}{L}\frac{\omega_2}{2}\frac{L^3}{3}\right]$, or

$$k(L-L_0) = \omega^2 L(m + \frac{m_0}{3}). \tag{1}$$

III. EXACT CALCULATION

The estimate turns out to be correct for small m_0, but we don't know what "small" means. Relax the assumption that the spring stretches uniformly, as fits the intuition that the parts of the spring with smaller r will be under greater tension, and will therefore stretch more, than the parts of the spring with larger r. Retain the linearity of the spring. Consider

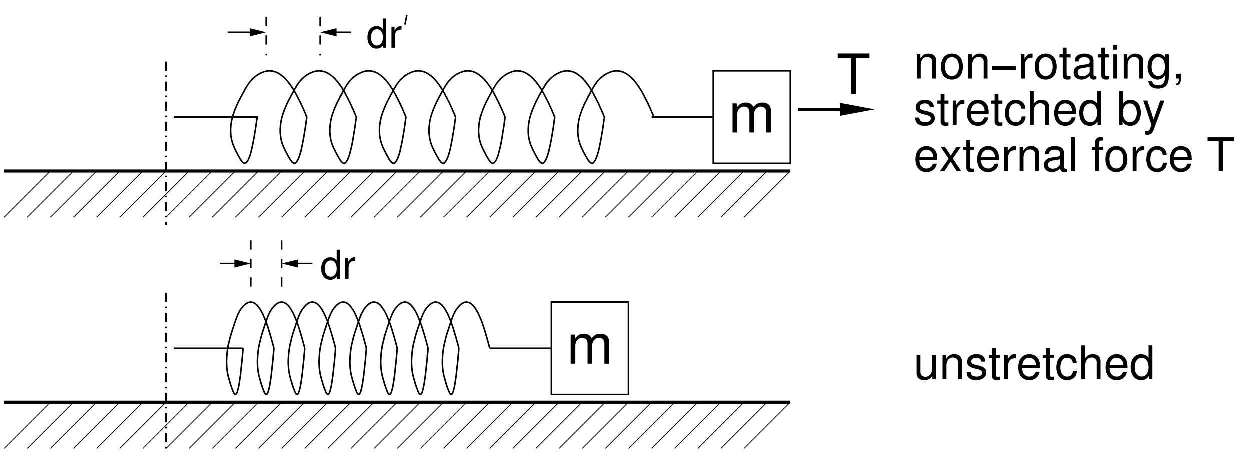

FIG. 2: Picture to go with Eq. 2.

a non-rotating spring stretched by an external force T, and write for an infinitesimal element

$$T = k(L - L_0) = kL_0\left(\frac{L}{L_0} - 1\right) = kL_0\left(\frac{dr'}{dr} - 1\right) = T. \tag{2}$$

The last inequality is also true for a rotating spring, in which case T is a function of position.

Now write radial force balance for an infinitesimal element of the rotating spring in equilibrium: $T(r' - \frac{dr'}{2}) - T(r' + \frac{dr'}{2}) = \frac{dm}{dr'}dr'\omega^2 r'$, or

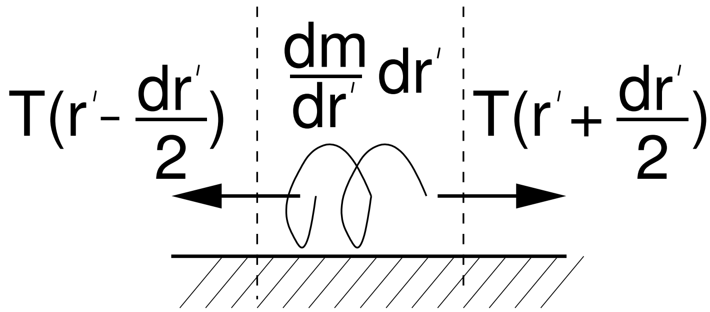

FIG. 3: Picture to go with Eq. 3.

$$-\frac{dT}{dr'} = \frac{dm}{dr'}\omega^2 r' \ .$$

When $\omega = 0$, the unstretched infinitesimal mass element has radial extent $dr = \frac{L_0}{m_0}dm$.
When $\omega \geq 0$, there is stretching: $dr' = \frac{dr'}{dr}dr = \left(\frac{T}{kL_0} + 1\right)dr \geq dr$.
Thus force balance is now

$$-\frac{dT}{dr'}\left(\frac{T}{kL_0} + 1\right) = \frac{m_0}{L_0}\omega^2 r' \ . \tag{3}$$

The tension at $r' = L$ is known: $T(L) = m\omega^2 L$. Thus Eq. 3 can be integrated to find $T(r')$. This results in a quadratic equation for $T(r')$,

$$\frac{T(r')^2}{2kL_0} + T(r') - \left[\frac{(m\omega^2 L)^2}{2kL_0} + m\omega^2 L\right] = \frac{m_0\omega^2}{2L_0}(L^2 - r'^2),$$

which can be solved to give

$$T(r') = -kL_0 + \left[(kL_0 + m\omega^2 L)^2 + km_0\omega^2(L^2 - r^2)\right]^{1/2}.$$

Note that $L = L(\omega)$; the problem isn't yet solved. However, since now $T(r')$ is known, the last equality in Eq. 2 can be written

$$\frac{dr}{L_0} = \frac{dr'}{T/k + L_0} = \frac{kdr'}{[(kL_0 + m\omega^2 L)^2 + km_0\omega^2 L^2 - km_0\omega^2 r'^2]^{1/2}},$$

and integrated.

$$\int_{inside}^{outside} \frac{dr}{L_0} = 1 = \int_0^L \frac{kdr'}{\sqrt{A - Br'^2}} \qquad \begin{array}{l} A \equiv (kL_0 + m\omega^2 L)^2 + km_0\omega^2 L^2 \\ B \equiv km_0\omega^2 \end{array} \tag{4}$$

After some algebra this gives

$$Q = q \cot q, \qquad \begin{array}{l} q \equiv \sqrt{\frac{m_0}{k}}\omega \\ Q \equiv \frac{L_0}{L} + \frac{m\omega^2}{k} \end{array} \tag{5}$$

which is a transcendental equation that gives L in terms of L_0, m_0, k, m, and ω.

IV. LIMITS

Consider Eq. 4 when $m_0 = 0$ (massless spring). Then we get

$$1 = \frac{L}{L_0 + \frac{m\omega^2}{k}L} \rightarrow L = \frac{L_0}{1 - \frac{m\omega^2}{k}},$$

which shows that $L \rightarrow \infty$ when $m\omega^2/k \rightarrow 1$, reflecting the fact that you can break a spring with this set-up by rotating the table too fast. If the spring isn't massless, this limit on ω decreases, so that the spring breaks when

$$\frac{q}{\tan q} = \frac{m\omega^2}{k}.$$

Even if there isn't any mass m attached the the spring, the spring will break when

$$\frac{q}{\tan q} = 0 \rightarrow q = \frac{\pi}{2},\ \ \omega = \frac{\pi}{2}\sqrt{\frac{k}{m_0}}.$$

V. RANGE OF APPLICABILITY OF ESTIMATE

The initial estimate required dr'/dr to be independent of r', and gave Eq. 1, which can be rewritten as $Q = 1 - q^2/3$. This can be compared with the exact result, Eq. 5:

$$Q = q \cot q \sim 1 - \frac{q^2}{3} - \frac{q^4}{45} - \frac{2q^2}{945} - \dots\ .$$

The difference between the two is small when the spring is not stretched much, and becomes increasingly important as the spring nears the breaking point, which comes at a smaller ω than the estimate predicts. For example, when $m_0/m = \pi/4$, the estimate predicts that the spring breaks at $\omega = 0.78895\sqrt{k/m_0}$, while the exact calculation gives $\omega = 0.78540\sqrt{k/m_0}$. Figure 4 summarizes the agreement between estimate and exact result for the range $0.1 \leq m_0/m \leq 10.0$: below the solid line, the estimated length of the spring is within 1% of the exact length; the spring breaks at the dashed line; in between the dashed line and the solid line the estimated length is less than the exact length by more than 1%.

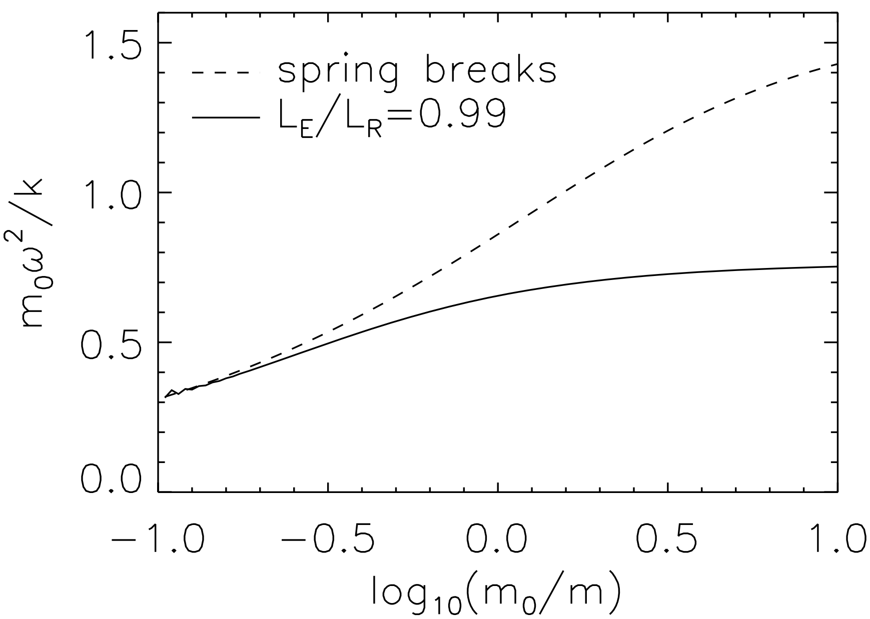

FIG. 4: Comparison of estimate and exact calculation. As ω is increased for a given m_0/m, the estimate increasingly underestimates the actual length of the spring. This underestimation reaches 1% at the solid line, and grows infinite at the dashed line.

VI. CONCLUSION

As shown in Fig. 4, as long as the spring is not very massive compared to the mass hooked to its end, the estimate does quite well, even if the spring is stretched to near breaking. To be directly relevant to Lab M6, the above analysis would have to be repeated to take into account the fact that the inner end of the spring is attached to a rigid support at $r_0 \neq 0$. This does not complicate the analysis, although the various numerical results would be changed. It seems clear that for the springs, masses, and angular frequencies relevant to lab M6, the estimate of the effect of spring-mass (Eq. 1) is quite probably sufficient, with other sources of error, such as non-linearity of the spring, dominating.