Last time…
Fields, forces, work, and potential

Electric forces and work

Electric potential energy
and electric potential

Potential from electric field

\[ dV = -\mathbf{E} \cdot d\mathbf{i} \]

\[ V = V_0 - |\mathbf{E}| l \]

\[ V = V_0 + |\mathbf{E}| l \]

 Equipotential lines

- Lines of constant potential
- In 3D, surfaces of constant potential

Topographic map

- Each lines is constant elevation
- Same as constant gravitational potential \( gh \) (energy = \( mgh \))
- Height interval between lines constant

Electric field from potential

- Said before that \( dW = F_{ext} \cdot d\mathbf{i} = -F_{Coulomb} \cdot d\mathbf{i} \)
- Spell out the vectors:
  \[ dV = -\mathbf{E} \cdot d\mathbf{i} \]
- This works for:
  \[ E_x = -\frac{dV}{dx}, \quad E_y = -\frac{dV}{dy}, \quad E_z = -\frac{dV}{dz} \]
- Usually written
  \[ \mathbf{E} = -\nabla V = \left( \frac{dV}{dx}, \frac{dV}{dy}, \frac{dV}{dz} \right) \]

Quick Quiz
Suppose the electric potential is constant everywhere. What is the electric field?

A) Positive
B) Negative
C) Increasing
D) Decreasing
E) Zero
Electric Potential - Uniform Field

\[ dV = -E \cdot ds \]
\[ \Rightarrow V_B - V_A = \int_A^B -E \cdot ds \]
\[ = \int_A^B E dx = -E \int_A^B (x_B - x_A) \]

Constant E-field corresponds to linearly decreasing (in direction of E) potential

Here V depends only on \( x \), not on \( y \)

Check of basic cases

- Previous quick quiz: uniform potential corresponds to zero electric field
  \[ E = 0 \]
  \[ V_B = V_A \]

- Linear potential corresponds to constant electric field
  \[ E = \frac{kQ}{r^2} \]
  \[ V_B = V_A \]

Potential (V) of spherical conductor

- What is V of spherical conductor relative to infinity?
- Charge on surface \( \Rightarrow \) spherical charge shell
- Gauss’ law \( \Rightarrow E = kQ / r^2 \) in the radial direction
- V is work / Coulomb to bring point charge from \( \infty \)

Quick quiz

Two conducting spheres of different radii connected by long conducting wire. What is approximately true of \( Q_1 \) and \( Q_2 \)?

- A) \( Q_2 > Q_1 \)
- B) \( Q_2 < Q_1 \)
- C) \( Q_2 = Q_1 \)

Connected spheres

- Since both must be at the same potential,
  \[ \frac{kQ_1}{R_1} = \frac{kQ_2}{R_2} \Rightarrow Q_2 = \frac{R_1}{R_2} Q_1 \]
  Smaller radius sphere has larger charge

- Surface charge densities?
  \[ \eta = \frac{Q}{4\pi R} \Rightarrow \eta_2 = \frac{R_1}{R_2} \eta_1 \]
  Smaller sphere has larger surface charge density

- Electric field?
  \[ E = \frac{\eta}{\varepsilon_0} \Rightarrow E_2 = \frac{R_1}{R_2} \eta_1 \]
  Local E-field bigger at more sharply curved (smaller R) regions

Varying E-fields on conductor

- Larger electric fields near smaller radii surfaces.
- Large electric fields at sharp points.
- Strong fields can ionize air atoms.
Potential and charge

- Have shown that a conductor has an electric potential, and that potential depends on its charge
- For a charged conducting sphere:
  \[ V(R) - V(\infty) = \frac{kQ}{R} \]

Quick Quiz

Consider this conducting object. When it has total charge \( Q_o \), its electric potential is \( V_o \). When it has charge \( 2Q_o \), its electric potential

A. is \( V_o \)
B. is \( 2V_o \)
C. is \( 4V_o \)
D. depends on shape

Capacitance

- Electric potential of any conducting object proportional to its total charge.
  \[ V = \frac{1}{C} Q \]
- \( C \) = capacitance
  - Large capacitance: need lots of charge to change potential
  - Small capacitance: small charge can change potential.

Capacitors

- Where did the charge come from?
  - Usually transferred from another conducting object, leaving opposite charge behind
- A capacitor consists of two conductors
  - Conductors generically called ‘plates’
  - Charge transferred between plates
    - Plates carry equal and opposite charges
    - Potential difference between plates proportional to charge transferred \( Q \)

Definition of Capacitance

- Same as for single conductor
  \[ \Delta V = \frac{1}{C} Q \]
- but \( \Delta V \) = potential difference between plates
  - \( Q \) = charge transferred between plates
- SI unit of capacitance is farad (F) = 1 Coulomb / Volt
- This is a very large unit: typically use
  - \( \mu F = 10^{-6} F \), \( nF = 10^{-9} F \), \( pF = 10^{-12} F \)

How was charge transferred?

- Battery has fixed electric potential difference across its terminals
- Conducting plates connected to battery terminals by conducting wires.
- \( \Delta V_{plates} = \Delta V_{battery} \) across plates
- Electrons move
  - from negative battery terminal to -\( Q \) plate
  - from +\( Q \) plate to positive battery terminal
- This charge motion requires work
  - The battery supplies the work
  \[ Q = C\Delta V \]
Work done to charge a capacitor

- Requires work to transfer charge $dq$ from one plate:
  $$dW = \Delta Vdq = \frac{q}{C}dq$$

- Total work = sum of incremental work
  $$W = \int \frac{q}{C}dq = \frac{Q}{2C}$$

- Work done stored as potential energy in capacitor
  $$U = \frac{Q^2}{2C} = \frac{Q}{2}CQ$$

Example: Parallel plate capacitor

- Charge $Q$ moved from right outer conductor to left conductor
- Charge only on inner surfaces
- Plate surfaces are charge sheets, each producing E-field
  $$E_{left} + E_{right} = \frac{\eta}{2\varepsilon_0} + \frac{\eta}{2\varepsilon_0} = \frac{\eta}{\varepsilon_0}$$
  Uniform field between plates

Quick Quiz

Electric field between plates of infinite parallel-plate capacitor has a constant value $\eta / \varepsilon_0$. What is the field outside of the plates?

A. $\eta / \varepsilon_0$
B. $\eta / 2 \varepsilon_0$
C. $- \eta / 2 \varepsilon_0$
D. $\eta / 4 \varepsilon_0$
E. 0

What is potential difference?

Potential difference $= V_+V_-$

$= \frac{-(-qEd)}{q}$

$\Delta V = Ed = \eta d / \varepsilon_0 = \frac{d}{(A \varepsilon_0)}$

$\Delta V = V_+ - V_- = \frac{d}{(A \varepsilon_0)}$

What is the capacitance?

$$\Delta V = V_+ - V_- = \frac{d}{(A \varepsilon_0)}$$

$$\Delta V = \frac{Q}{C}$$

$$C = \frac{\varepsilon_0 A}{d}$$

This is a geometrical factor

Energy stored in parallel-plate capacitor

$$U = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2} \frac{\varepsilon_0 A}{d} (Ed)^2 = \frac{1}{2} (Ad)\varepsilon_0 E^2$$

Energy density

$$U / (Ad) = \frac{1}{2} \varepsilon_0 E^2$$