Chapter 11 – Gravity
Lecture 2

Physics 201
Fall 2009

• The Cavendish experiment (second try)
• Gravitational potential energy
• Escape velocity
• Gravitational Field of a point mass
• Gravitational Field for mass distributions
  – Discrete
  – Rod
  – Spherical shell
  – Sphere
• Gravitational potential energy of a system of particles
• Black holes

Measuring G

• G was first measured by Henry Cavendish in 1798
• The apparatus shown here allowed the attractive force between two spheres to cause the rod to rotate
• The mirror amplifies the motion
• It was repeated for various masses

\[ \text{Magnitude} = G \frac{mM}{r^2} \]

\[ G = 6.67 \times 10^{-11} \text{ N m}^2 / \text{kg}^2 \]
A woman whose weight on Earth is 500 N is lifted to a height of two Earth radii above the surface of Earth. Her weight decreases to one-half of the original amount. It can also decrease to one-quarter of the original amount, or it could remain unchanged. It might also decrease to one-third of the original amount. Alternatively, it could decrease to one-ninth of the original amount.

A woman whose weight on Earth is 500 N is lifted to a height of two Earth radii above the surface of Earth. Her weight

A. decreases to one-half of the original amount.
B. decreases to one-quarter of the original amount.
C. does not change.
D. decreases to one-third of the original amount.
E. decreases to one-ninth of the original amount.
From work to gravitational potential energy.

In the example before, it does not matter on what path the person is elevated to 2 Earth radii above. Only the final height (or distance) matters for the total amount of work performed.

\[
\begin{align*}
\mathbf{F}_g &= -G \frac{mM}{r^2} \hat{r}_{\text{final}} \\
&= -G \frac{mM}{r^2} \hat{r} \\
dr &= d \ell \cos \phi = \hat{r} \cdot d \ell
\end{align*}
\]

Potential Energy

Force (1,2) = \(-G \frac{mM}{r^2} \hat{r}_{1,2}\) \(G = 6.67 \times 10^{-11} \text{ N m}^2 / \text{kg}^2\)

Work done to bring mass \(m\) from initial to final position.

\[
PE = -W = - \int \mathbf{F} \cdot d \mathbf{r} = - \int \left( -G \frac{mM}{r^2} \right) dr \\
= \left. GmM \left( \frac{1}{r_f} - \frac{1}{r_i} \right) \right|_i^{f} \\

\]

Zero point is arbitrary. Choose zero at infinity.
Gravitational potential energy

\[ U(r) = -G \frac{Mm}{r} \]

\[
\begin{align*}
U(r) & = 0 \\
U(R_E) & = -\frac{GM_Em}{R_E} = -mgR_E \\
U(R_E + h) & \\
U(R_E) + mgh & \\
U(R_E) + mgh & \\
r & \\
0 & \\
\end{align*}
\]
Escaping Gravity

E>0: object is not bound
E<0: object is bound to gravity. Field
E=0: kinetic energy just enough to escape gravity (K=U)

11/17/09

Kinetic energy of the object must be greater than its gravitational potential energy
– This defines the minimum velocity to escape

KE+PE = constant
– Consider case when speed is just sufficient to escape to infinity with vanishing final velocity
– At infinity, KE+PE=0, therefore, on Earth,

\[ \frac{1}{2} m v_{esc}^2 - G \frac{M_E m}{R_E} = 0 \Rightarrow \]

\[ v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = 11.2 \text{ km/s} = 25000 \text{ mph} \]
Al Shepard, 36 years ago, February 1971

Al Shepard plays golf on the moon.

Video clip of this shot at: http://www.hq.nasa.gov/alsj/frame.html

Radio transcript from Apollo 14

- 135:09:01 Haise: That looked like a slice to me, Al
- 135:09:03 Shepard: Here we go. Straight as a die; one more. (Long Pause)[Al's third swing finally connects and sends the ball off-camera to the right, apparently on a fairly low trajectory. He drops a second ball, which rolls left and toward the TV camera. Al gets himself in position and connects again. The trajectory of this shot appears to be similar to the previous one.]
- 135:09:20 Shepard: Miles and miles and miles.
- 135:09:26 Haise: Very good, Al.
Quiz

• You are on the moon and you know how to calculate the escape velocity:
  \[ v_{esc} = \sqrt{\frac{2GM}{R}} \]

• You find that it is 2.37km/s

A projectile from the moon surface will escape even if it is shot horizontally, not vertically with a speed of at least 2.37km/s

A) Correct
B) Not correct

Gravity near Earth’s surface...

• Near the Earth’s surface:
  – \( R_{12} = R_E \)
  – Won’t change much if we stay near the Earth’s surface.
    – since \( R_E \gg h, R_E + h \sim R_E \).

\[ \left| F_g \right| = G \frac{M_E m}{R_E^2} \]
Gravity...

- Near the Earth’s surface...
  \[ |F_g| = G \frac{M_E m}{R_E^2} = m \left( G \frac{M_E}{R_E^2} \right) \]
  \[ = g \]

- So \(|F_g| = mg = ma\)
  - \(a = g\) \(\text{All objects accelerate with acceleration } g, \text{ regardless of their mass!}\)

Where:
\[ g = G \frac{M_E}{R_E^2} = 9.81 \text{ m/s}^2 \]

Gravity near Earth’s surface...

\[ U(r) = -\frac{GM_Em}{r} \]

\[ U(R_E) = -\frac{GM_Em}{R_E} = -mgR_E \]
Variation of $g$ with Height

This is twice the Earth radius: 6000km

We know $F$ should drop with $r^2$

Indeed, "g" has dropped to 9.81/4 m/s²

<table>
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<th>Altitude $h$ (km)</th>
<th>$g$ (m/s²)</th>
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<tr>
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<td>0.13</td>
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<tr>
<td>$\infty$</td>
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Question

Suppose you are standing on a bathroom scale in your dorm room and it says that your weight is $W$. What will the same scale say your weight is on the surface of the mysterious Planet X?

You are told that $R_X \sim 20 R_{\text{Earth}}$ and $M_X \sim 300 M_{\text{Earth}}$.

(a) $0.75 W$
(b) $1.5 W$
(c) $2.25 W$

$$F_g = G \frac{mM}{r^2}$$

$$F_{g,X} = \frac{300}{20^2} F_{g,\text{Earth}}$$
Gravitational Field

- Gravitational force:
  \[ \vec{F}_{12} = G \frac{m_1 m_2}{r_{12}^2} \hat{r}_{12} \]

- Definition of the gravitational field that will act on any masspoint:
  \[ \frac{\vec{g}}{g} = \frac{\vec{F}_g}{m} \]

- If the field is caused by a mass distribution we need to sum over all masspoints:
  \[ g = \sum g_i \frac{F_i}{m} \]

Gravitational field

- The gravitational field vectors point in the direction of the acceleration a particle would experience if placed in that field.
- The magnitude is that of the freefall acceleration at that location.
- The gravitational field describes the "effect" that any object has on the empty space around itself in terms of the force that would be present if a second object were somewhere in that space.

\[ \vec{g} = \frac{\vec{F}_g}{m} = -\frac{G M}{r^2} \hat{r} \]

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Gravitational Field

Two mass points, fieldpoint in plane of symmetry

\[ g = \sum g_i \]

Magnitude of field due to each mass:

\[ g_1 = g_2 = G \frac{M}{r^2} \]

Need to add \( x \) and \( y \) component of \( g_1 \) and \( g_2 \)

\[ x \]-component:

\[ g_x = g_{1x} + g_{2x} = 2G \frac{m}{r^2} \cos(\theta) = 2G \frac{m}{r^2} \frac{x_p}{r} = 2G \frac{mx_p}{r^3} \]

\( y \)-component is zero for symmetry reasons

\[ g_y = 0 \]

Gravitational Field

Field due to rod of length \( L \) on a point along its axis.

Field by one mass element \( dm \):

\[ dg = -G \frac{dm}{r^2} \]

Integrate over all mass elements \( dm \):

\[ g = \int dg = \int -G \frac{dm}{r^2} = - \int \frac{GM}{r} \frac{dx}{L \left(x_p - x_s\right)} = \frac{GM}{x_p^2 - \left(L/2\right)^2} \]
**Gravitational Field**

Field due to spherical symmetric mass distribution, a shell of mass $M$ and radius $R$:

Field of a spherical shell

\[
\begin{align*}
-\mathbf{g} &= -G \frac{M}{r^2} \mathbf{r} & r > R \\
-\mathbf{g} &= 0 & r < R
\end{align*}
\]

Geometry: spherical shell is 0 anywhere inside

**Gravitational Field**

Field due to homogeneous massive sphere

Field inside the sphere

\[
g = -G \frac{M}{R^3} r & \quad r < R
\]
Gravitational Potential Energy

- For any two particles, the gravitational potential energy function becomes
  \[ U = -\frac{Gm_1m_2}{r} \]

- The gravitational potential energy between any two particles varies as \(1/r\)
  - Remember the force varies as \(1/r^2\)
- The potential energy is negative because the force is attractive and we chose the potential energy to be zero at infinite separation

- An external agent must do positive work to increase the separation between two objects
  - The work done by the external agent produces an increase in the gravitational potential energy as the particles are separated
  - \(U\) becomes less negative

Binding Energy

- The absolute value of the potential energy can be thought of as the binding energy
- If an external agent applies a force larger than the binding energy, the excess energy will be in the form of kinetic energy of the particles when they are at infinite separation
Systems with Three or More Particles

- The total gravitational potential energy of the system is the sum over all pairs of particles.
- Gravitational potential energy obeys the superposition principle.
- Each pair of particles contributes a term of $U$.
- The absolute value of $U_{\text{total}}$ represents the work needed to separate the particles by an infinite distance.

$$U_{\text{total}} = U_{12} + U_{13} + U_{23}$$
$$= -G \left( \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right)$$

Potential energy of a system of masses

- What is the total potential energy of this mass system?

$$U = -3G \frac{mm}{L}$$
Four identical masses, each of mass $M$, are placed at the corners of a square of side $L$. The total potential energy of the masses is equal to $-xGM^2/L$, where $x$ equals

A. 4
B. $4 + 2\sqrt{2}$
C. $4 + \sqrt{2}$
D. $4 + \frac{1}{\sqrt{2}}$
E. $2 + 2\sqrt{2}$
Black Holes

- **A black hole** is the remains of a star that has collapsed under its own gravitational force
- The escape speed for a black hole is very large due to the concentration of a large mass into a sphere of very small radius
  - If the escape speed exceeds the speed of light, radiation cannot escape and it appears black
- The critical radius at which the escape speed equals $c$ is called the **Schwarzschild radius**, $R_S$
- The imaginary surface of a sphere with this radius is called the **event horizon**
  - This is the limit of how close you can approach the black hole and still escape

Black Holes and Accretion Disks

- Although light from a black hole cannot escape, light from events taking place near the black hole should be visible
- If a binary star system has a black hole and a normal star, the material from the normal star can be pulled into the black hole
  - This material forms an **accretion disk** around the black hole
- Friction among the particles in the disk transforms mechanical energy into internal energy
- The orbital height of the material above the event horizon decreases and the temperature rises
- The high-temperature material emits radiation, extending well into the x-ray region
  - These x-rays are characteristics of black holes
Black Holes at Centers of Galaxies

- There is evidence that supermassive black holes exist at the centers of galaxies (M=100 million solar masses)
- Theory predicts jets of materials should be evident along the rotational axis of the black hole

An Hubble Space Telescope image of the galaxy M87. The jet of material in the right frame is thought to be evidence of a supermassive black hole at the galaxy's center.