Physics 201: 
Chapter 14 – Oscillations

Dec 8, 2009

- Hooke’s Law
- Spring
- Simple Harmonic Motion
- Energy
Springs

- **Hooke’s Law:** The force exerted by a spring is proportional to the distance the spring is stretched or compressed from its relaxed position.

\[ F_x = -kx \]

Where \( x \) is the displacement from the relaxed position and \( k \) is the constant of proportionality.

(often called “spring constant”)

![Diagram of Hooke's Law](image)
We know that if we stretch a spring with a mass on the end and let it go, the mass will oscillate back and forth (if there is no friction).

This oscillation is called **Simple Harmonic Motion**, and is actually easy to understand...
At any given instant we know that $F = ma$ must be true.

But in this case $F = -kx$

and $ma = \frac{d^2 x}{dt^2}$

So: $-kx = ma = m \frac{d^2 x}{dt^2}$
SHM Dynamics...

Define 
\[ \omega = \sqrt{\frac{k}{m}} \]

Where \( \omega \) is the angular frequency of motion

Try the solution \( x = A \cos(\omega t) \)

\[ \frac{dx}{dt} = -\omega A \sin(\omega t) \]

\[ \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t) = -\omega^2 x \]

This works, so it must be a solution!
SHM Dynamics...

- We just showed that \( \frac{d^2x}{dt^2} = -\omega^2 x \) (from \( F = ma \)) has the solution \( x = A \cos(\omega t) \) but \( x = A \sin(\omega t) \) is also a solution.

- The most general solution is a linear combination (sum) of these two solutions!

\[
x = B \sin(\omega t) + C \cos(\omega t) = A \cos(\omega t + \phi)
\]

\[
\frac{dx}{dt} = \omega B \cos(\omega t) - \omega C \sin(\omega t)
\]

\[
\frac{d^2x}{dt^2} = -\omega^2 B \sin(\omega t) - \omega^2 C \cos(\omega t) = -\omega^2 x \quad \text{ok}
\]
SHM Dynamics...

\[ x = A \cos(\omega t + \phi) \]

- But wait a minute...what does *angular frequency* \( \omega \) have to do with moving back & forth *in a straight line*??

The spring’s motion with amplitude \( A \) is identical to the \( x \)-component of a particle in uniform circular motion with radius \( A \).
Example

- A mass $m = 2 \text{ kg}$ on a spring oscillates with amplitude $A = 10 \text{ cm}$. At $t = 0$ its speed is maximum, and is $v = +2 \text{ m/s}$.
  - What is the angular frequency of oscillation $\omega$?
  - What is the spring constant $k$?

$$v_{\text{MAX}} = \omega A$$

$$\omega = \frac{v_{\text{MAX}}}{A} = \frac{2 \text{ m/s}}{10 \text{ cm}} = 20 \text{ s}^{-1}$$

Also:

$$\omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad k = m\omega^2$$

So $k = (2 \text{ kg}) \times (20 \text{ s}^{-1})^2 = 800 \text{ kg/s}^2 = 800 \text{ N/m}$
Initial Conditions

Use “initial conditions” to determine phase $\phi$!

Suppose we are told $x(0) = 0$, and $x$ is initially increasing (i.e. $v(0) = \text{positive}$):

$x(0) = 0 = A \cos(\phi) \quad \Rightarrow \quad \phi = \pi/2 \text{ or } -\pi/2$
$v(0) > 0 = -\omega A \sin(\phi) \quad \Rightarrow \quad \phi < 0$

So $\phi = -\pi/2$

$x(t) = A \cos(\omega t + \phi)$
$v(t) = -\omega A \sin(\omega t + \phi)$
$a(t) = -\omega^2 A \cos(\omega t + \phi)$
Initial Conditions...

So we find $\phi = -\pi/2$

$$x(t) = A \cos(\omega t - \pi/2)$$
$$v(t) = -\omega A \sin(\omega t - \pi/2)$$
$$a(t) = -\omega^2 A \cos(\omega t - \pi/2)$$

$$x(t) = A \sin(\omega t)$$
$$v(t) = \omega A \cos(\omega t)$$
$$a(t) = -\omega^2 A \sin(\omega t)$$
What about Vertical Springs?

- We already know that for a vertical spring if $y$ is measured from the equilibrium position.

- The force of the spring is the negative derivative of this function:
  $$U = \frac{1}{2}ky^2$$

- So this will be just like the horizontal case:
  $$-ky = ma = m\frac{d^2y}{dt^2}$$

Which has solution $y = A \cos(\omega t + \phi)$

where $\omega = \sqrt{\frac{k}{m}}$
Recall that the torque due to gravity about the rotation \((z)\) axis is

\[ \tau = -mgd \]

\[ d = L\sin \theta \approx L\theta \] for small \(\theta\)

so \[ \tau = -mgL\theta \]

But \(\tau = I\alpha\), where \(I = mL^2\)

\[ \frac{d^2\theta}{dt^2} = -\omega^2\theta \]

where \[ \omega = \sqrt{\frac{g}{L}} \]

Differential equation for simple harmonic motion!

\[ \theta = \theta_0 \cos(\omega t + \varphi) \]
You are sitting on a swing. A friend gives you a small push and you start swinging back & forth with period $T_1$.

Suppose you were standing on the swing rather than sitting. When given a small push you start swinging back & forth with period $T_2$.

Which of the following is true:

(a) $T_1 = T_2$

(b) $T_1 > T_2$

(c) $T_1 < T_2$
Solution

- We have shown that for a simple pendulum \[ \omega = \sqrt{\frac{g}{L}} \]

Recall the pendulum depended on the torque from gravity. Now the moment arm is from the center of mass and the oscillation point so \( L \) is shorter

Since \[ T = \frac{2\pi}{\omega} \] \[ \Rightarrow \] \[ T = 2\pi \sqrt{\frac{L}{g}} \]

- If we make a pendulum shorter, it oscillates faster (shorter period)
Simple Harmonic Motion: Summary

Force: \( \frac{d^2 s}{dt^2} = -\omega^2 s \)

Solution: \( s = A \cos(\omega t + \phi) \)

\( \omega = \sqrt{\frac{k}{m}} \)

\( \omega = \sqrt{\frac{g}{L}} \)
Energy for simple harmonic motion: $E = K + U$

\[ x = A \cos(\omega t) \]

\[ U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \cos^2(\omega t) \]

\[ v = \frac{dx}{dt} = -\omega A \sin(\omega t) \]

\[ K = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t) \]

\[ E = K + U = K_{Max} = \frac{1}{2} m\omega^2 A^2 = U_{Max} = \frac{1}{2} k^2 A^2 \]
The potential energy function

\[ U = \frac{1}{2} k x^2 \]

\[ K = \frac{1}{2} m v^2 \]

\[ E = U + K \]

\[ = \frac{1}{2} k A^2 = \frac{1}{2} m v_{\text{max}}^2 \]
Energy in simple harmonic motion

- For both the spring and the pendulum, we can derive the SHM solution by using energy conservation.

- The total energy \((K + U)\) of a system undergoing SHM will always be constant!

- This is not surprising since there are only conservative forces present, hence \(K+U\) energy is conserved.
Energy conservation

\[ E_{\text{total}} = \frac{1}{2} M v^2 + \frac{1}{2} k x^2 = \text{constant} \]

\[ KE_{\text{max}} = M v_{\text{max}}^2 / 2 = M \omega^2 A^2 / 2 = k A^2 / 2 \]

\[ PE_{\text{max}} = k A^2 / 2 \]

\[ E_{\text{total}} = k A^2 / 2 \]
The harmonic oscillator is often a very good approximation for (non harmonic) oscillations with small amplitude.

\[ U \approx \frac{1}{2} kx^2 \]

for small \( x \)
Suppose we have some arbitrarily shaped solid of mass $M$ hung on a fixed axis, and that we know where the CM is located and what the moment of inertia $I$ about the axis is.

The torque about the rotation ($z$) axis for small $\theta$ is (using $\sin \theta \approx \theta$)

$$\tau = -Mgd \approx -MgR\theta$$

$$-MgR\theta = I \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\omega^2 \theta$$ \text{ where } \omega = \sqrt{\frac{MgR}{I}}

$$\theta = \theta_0 \cos(\omega t + \phi)$$
Consider an object suspended by a wire attached at its CM. The wire defines the rotation axis, and the moment of inertia $I$ about this axis is known.

The wire acts like a “rotational spring.”

- When the object is rotated, the wire is twisted. This produces a torque that opposes the rotation.
- In analogy with a spring, the torque produced is proportional to the displacement: $\tau = -k\theta$
Torsional Oscillator…

- Since $\tau = -k\theta$, $\tau = I\alpha$ becomes

$$-k\theta = I \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\omega^2 \theta \quad \text{where} \quad \omega = \sqrt{\frac{k}{I}}$$

This is similar to the “mass on spring” except $I$ has taken the place of $m$ (no surprise).
Damped Oscillations

- In many real systems, nonconservative forces are present
  - The system is no longer ideal
  - Friction is a common nonconservative force
- In this case, the mechanical energy of the system diminishes in time, the motion is said to be damped

- The amplitude decreases with time
- The blue dashed lines on the graph represent the envelope of the motion
One example of damped motion occurs when an object is attached to a spring and submerged in a viscous liquid.

The retarding force can be expressed as \( R = -b \, v \) where \( b \) is a constant.

\( b \) is called the **damping coefficient**.

The restoring force is \(-kx\).

From Newton’s Second Law

\[ \Sigma F_x = -k \, x - b \, v_x = ma_x \]

or,

\[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0 \]
When $b$ is small enough, the solution to this equation is:

$$x = A_0 e^{-(b/2m)t} \cos(\omega' t + \delta)$$

with

$$\omega' = \omega_0 \sqrt{1 - \left(\frac{b}{2m\omega_0}\right)^2}, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

(This solution applies when $b < 2m\omega_0$.)
**Damping Oscillation, Example**

- The position can be described by
  
  \[ x = A_0 e^{-\left(\frac{b}{2m}\right)t} \cos(\omega' t + \delta) \]

- The angular frequency is
  
  \[ \omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \]

- When the retarding force is small, the oscillatory character of the motion is preserved, but the amplitude decreases exponentially with time
- The motion ultimately ceases
The energy of a weakly damped harmonic oscillator decays exponentially in time.

The potential energy is $\frac{1}{2}kx^2$:

$$x^2 = \left( A_0 e^{-(b/2m)t} \cos(\omega't + \delta) \right)^2$$

$$= A_0^2 e^{-(b/m)t} \cos^2(\omega't + \delta)$$

Average over a period when $(b/m) << \omega'$:

Average of $\cos^2 = \frac{1}{2}$, so averaged over a period,

$$\langle x^2 \rangle \approx A_0^2 e^{-(b/m)t}.$$
Energy of weakly damped harmonic oscillator:

- The energy of a weakly damped harmonic oscillator decays exponentially in time.

The kinetic energy is \( \frac{1}{2}mv^2 \):

\[
v^2 = \left( \frac{d}{dt} \left( A_0 e^{-(b/2m)t} \cos(\omega't + \delta) \right) \right)^2
\]

\[
= \left( -(b/2m)A_0 e^{-(b/2m)t} \cos(\omega't + \delta) - A_0 e^{-(b/2m)t} \sin(\omega't + \delta) \right)^2
\]

Average energy over a period when \((b/m)<<\omega'\):

Average of \(\cos^2 = \text{average of } \sin^2 = \frac{1}{2}\), and average of \(\sin x \cos x = 0\), so averaged over a period,

\[
\left< v^2 \right> \propto e^{-(b/m)t}.
\]
Energy of weakly damped harmonic oscillator:

- The energy of a weakly damped harmonic oscillator decays exponentially in time.

Both kinetic and potential energy (averaged over a period) decay as $e^{-(b/m)t}$.
So total energy is proportional to $e^{-(b/m)t}$. 
Types of Damping

\[ \omega_0 = \sqrt{\frac{k}{m}} \]

- \(\omega_0\) is also called the *natural frequency* of the system.
- If \(R_{\text{max}} = b v_{\text{max}} < kA\), the system is said to be *underdamped*.
- When \(b\) reaches a critical value \(b_c\) such that \(b_c / 2m = \omega_0\), the system will not oscillate.
  - The system is said to be *critically damped*.
- If \(R_{\text{max}} = b v_{\text{max}} > kA\) and \(b/2m > \omega_0\), the system is said to be *overdamped*.
- For critically damped and overdamped there is no angular frequency.
**Forced Oscillations**

- It is possible to compensate for the loss of energy in a damped system by applying an external sinusoidal force (with angular frequency \( \omega \))
- The amplitude of the motion remains constant if the energy input per cycle exactly equals the decrease in mechanical energy in each cycle that results from resistive forces
- After a driving force on an initially stationary object begins to act, the amplitude of the oscillation will increase
- After a sufficiently long period of time, \( E_{\text{driving}} = E_{\text{lost to internal}} \)
  - Then a steady-state condition is reached
  - The oscillations will proceed with constant amplitude

- **The amplitude of a driven oscillation is**

\[
A = \frac{F_0}{m} \sqrt{\left( \frac{b \omega}{m} \right)^2 + \left( \omega - \omega_0 \right)^2}
\]

- \( \omega_0 \) is the natural frequency of the undamped oscillator
Resonance

- When $\omega \gg \omega_0$ an increase in amplitude occurs
- This dramatic increase in the amplitude is called **resonance**
- The natural frequency $\omega_0$ is also called the resonance frequency
- At resonance, the applied force is in phase with the velocity and the power transferred to the oscillator is a maximum
  - The applied force and $v$ are both proportional to $\sin(\omega t + \phi)$
  - The power delivered is $F \cdot v$
    - This is a maximum when $F$ and $v$ are in phase
- Resonance (maximum peak) occurs when driving frequency equals the natural frequency
- The amplitude increases with decreased damping
- The curve broadens as the damping increases
- The shape of the resonance curve depends on $b$
The amplitude of a system moving with simple harmonic motion is doubled. The total energy will then be

- 4 times larger
- 2 times larger
- the same as it was
- half as much
- quarter as much

\[ U = \frac{1}{2} kx^2 + \frac{1}{2} mv^2 \]

at \( x = A, v = 0 \)

\[ U = \frac{1}{2} kA^2 \]
A glider of mass $m$ is attached to springs on both ends, which are attached to the ends of a frictionless track. The glider moved by 0.2 m to the right, and let go to oscillate. If $m = 2$ kg, and spring constants are $k_1 = 800$ N/m and $k_2 = 500$ N/m, the frequency of oscillation (in Hz) is approximately

- $6$ Hz
- $2$ Hz
- $4$ Hz
- $8$ Hz
- $10$ Hz

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{k / m}}{2\pi} = \frac{\sqrt{800 + 500}}{2\pi} = \frac{\sqrt{1300}}{2\pi} = 4 \text{ Hz}$$
A 810-g block oscillates on the end of a spring whose force constant is 60 N/m. The mass moves in a fluid which offers a resistive force proportional to its speed – the proportionality constant is 0.162 N.s/m.

- Write the equation of motion and solution.

- What is the period of motion? 0.730 s

- What is the fractional decrease in amplitude per cycle? 0.070

- Write the displacement as a function of time if at \( t = 0, \ x = 0 \) and at \( t = 1 \) s, \( x = 0.120 \) m.

Solution: \( x(t) = 0.182e^{-0.100t} \sin(8.61t + \phi) \)