EXAM 3

Please print your name and section number (or TA’s name) clearly on all pages. Show all your work in the space immediately below each problem. Your final answer must be placed in the boxes provided. Problems will be graded on reasoning and intermediate steps as well as on the final answer. Be sure to include units wherever necessary, and the direction of vectors. Each problem is worth 20 points. Try to be neat! Check your answers to see that they have the correct dimensions (units) and are the right order of magnitude. You are allowed one sheet of notes (8.5” x 11”, 2 sides), a calculator, and the constants in this exam booklet. The exam lasts exactly 90 minutes.

Constants:
Avogadro’s Number: \( N_A = 6.02 \times 10^{23} \) molecules/mole

<table>
<thead>
<tr>
<th>Material</th>
<th>Density, ( \rho ) (kg/m(^3))</th>
<th>Young’s Modulus, ( Y ) (10(^9) N/m(^2))</th>
<th>Ultimate (Breaking) Strength (compression) (10(^6) N/m(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>7860</td>
<td>200</td>
<td>400</td>
</tr>
<tr>
<td>Bone</td>
<td>1900</td>
<td>9</td>
<td>170</td>
</tr>
<tr>
<td>Wood (Douglas fir)</td>
<td>525</td>
<td>13</td>
<td>50</td>
</tr>
</tbody>
</table>

Mathematics:

- trigonometric identity: \( \sin A + \sin B = 2\cos[(A-B)/2]\sin[(A+B)/2] \)

- small angle approximations: \( \sin \theta = \theta - \theta^3/3 + O(\theta^5) \)

\( \cos \theta = 1 - \theta^2/2 + O(\theta^4) \)

SCORE:

Problem 1: ________
Problem 2: ________
Problem 3: ________
Problem 4: ________
Problem 5: ________
TOTAL: ________

Don't open the exam until you are instructed to start.

“Success is not final, failure is not fatal: it is the courage to continue that counts.” Winston Churchill
PROBLEM 1

a.) The figure shows a safe of mass $M = 430 \text{ kg}$, hanging by a rope from a boom with dimensions $a = 1.9 \text{ m}$ and $b = 2.5 \text{ m}$. The boom consists of a hinged beam and a horizontal cable. The uniform beam has a mass of $m = 85 \text{ kg}$; the masses of the cable and rope are negligible. What is the tension, $T_c$, in the cable? (8 pts)

\[ \sum F = 0 \text{ and } \sum \vec{E} = 0 \text{ about any point.} \]

Let's compute $\sum \vec{E}$ about the hinge:

\[ T_c a - mg \frac{b}{2} - Mg b = 0 \]

\[ T_c = \frac{gb}{a} \left( \frac{m}{2} + M \right) \]

Sensible? If $a < b$ then $T_c$ gets big.

\[ T_c = 6100 \text{ N} \]

b. The deep-sea research diving vessel Alvin can operate at depths of 4500 m. With its ballast tanks empty, the vessel displaces a total volume of 16.8 m³ and has a total mass of 17,000 kg. What minimum volume must the ballast tanks have to permit the vessel to dive? (Ballast tanks are containers within the vessel to which water is added or removed. They are used to increase or decrease the mass of the vessel and hence make it sink or float.) (6 pts)

\[ \rho_{sea\,water} = 1025 \text{ kg/m}^3 \]

Bouyant force $B = V_{\text{displaced}} \rho_{water} g = 17,220 \text{ kg} \cdot \text{g}$

Weight force $w = 17,000 \text{ kg} \cdot \text{g}$

so, tank need to add 220 kg of mass, so $V_{\text{tank}} = \frac{220}{1025} = 0.21 \text{ m}^3$
c. Water circulates throughout a house in a hot-water heating system. If the water is pumped at a speed of 0.5 m/s through a 4.0 cm diameter pipe in the basement at a pressure of 3.0 atm, what will be the water speed and pressure in a 2.6 cm diameter pipe on the second floor, 5.0 m above the basement? Assume the pipes do not divide into branches. The density of water is 1000 kg/m³. (6 pts)

\[ \nu_{up} = \frac{\nu_B}{\nu_{up}} \times \frac{\pi (0.020)^2}{\pi (0.013)^2} = 1.2 \text{ m/s} \]  
Continuity equation.

\[ \rho + \frac{1}{2} \rho \nu^2 + \rho g h = \text{constant} \]

Barnes: Eq'n.

\[ \rho_B + \frac{1}{2} \rho \nu_B^2 + \phi = \rho_{up} + \frac{1}{2} \rho \nu_{up}^2 + \rho g h \]

\[ \rho_{up} = 3 \times 10^3 \times 3 \rho_a + \frac{1}{2} \rho \left( \nu_B^2 - \nu_{up}^2 \right) - \rho g \times 5.0 \]

\[ 1000 \]

\[ 0.5 \]

\[ 3 \times 10^3 \]

\[ 7.81 \]

\[ \frac{2 \nu_B^2}{1 - \left( \frac{\nu_{up}}{\nu_B} \right)^2} \]

\[ 1 - \left( \frac{0.020}{0.013} \right)^4 \]

\[ = 3.03 \times 10^3 \]

\[ - 575 \]

\[ - 49,050 \]

\[ = 2.5 \times 10^3 \rho_a = 2.5 \text{ atm} \]
Problem 2

A block of mass $M$ hangs initially at rest from the ceiling through a string of length $L$. A bullet of mass $m$ with initial velocity $v_i$ in the horizontal direction strikes the block. If the mass is swinging, the string makes an angle $\theta(t)$ (a function of time) with respect to the vertical. After the bullet lodges in the block, the maximum angle $\theta_{\text{max}}$ that the string makes with respect to the vertical (assuming $\theta_{\text{max}} \ll 1$) is

$$
\theta_{\text{max}} = \frac{v_i m}{M + m} \sqrt{\frac{1}{Lg}}.
$$

a) What is the differential equation governing $\theta(t)$ after the bullet lodges (assuming $\theta(t) \ll 1$)? (10 pts)

ANS: As shown in lecture, the horizontal position equation is

$$
-T \sin \theta = m\ddot{x}
$$

$$
-T \theta \approx mL\ddot{\theta}
$$

where $T$ is the tension and have used $\sin \theta \approx \theta$ for $\theta \ll 1$. As shown in lecture, we also have

$$
T \cos \theta - mg = m\frac{d^2}{dt^2}L(1 - \cos \theta)
$$

$$
[T + \mathcal{O}(\theta^2)] - mg \approx mL\frac{d^2}{dt^2}[0 + \mathcal{O}(\theta^2)]
$$

$$
T - mg \approx 0 + \mathcal{O}(\theta^2)
$$

where we have used $\cos \theta = 1 - \frac{1}{2} \theta^2 + ...$ Hence, we find

$$
\ddot{\theta} = -\frac{g}{L}\theta.
$$

(1)
b) If $\theta(t)$ (the solution to part a) is written as

$$\theta(t) = A \cos(w t + \phi),$$

what is $A$ and $w$? Express your answers in terms of $\{v_i, m, M, g, L\}$? [Hint: This problem requires you to know the answer to part a). If you do not know the answer to part a), say how you would obtain $w$ had you known the answer to part b).] (5 pts)

**ANS:** By definition, we have

$$A = \theta_{\text{max}} = v_i \frac{m}{M + m} \sqrt{\frac{1}{Lg}}$$

Plugging in $\theta = A \cos(w t + \phi)$, into Eq. (1) gives

$$w = \sqrt{\frac{g}{L}}.$$ 

c) If wind drag slows the mass down such that

$$\vec{F}_{\text{drag}} = -b \vec{\theta}$$

what is the new differential equation for $\theta(t)$? (5 pts)

**ANS:** Writing Newton’s equation of motion in the horizontal direction just as in part a), we have

$$mL \ddot{\theta} = -mg \theta - bL \dot{\theta}$$

Hence,

$$\ddot{\theta} = \frac{-g}{L} \theta - \frac{b}{m} \dot{\theta}.$$
Problem 3

a. Write down an equation, \( y(x,t) \), for a transverse standing wave on a string of length \( L = 0.20 \) m that is fixed at each end. The standing wave has wavelength \( \lambda = 0.20 \) m, frequency \( f = 400 \) Hz, and amplitude \( A = 4 \) cm. \((5 \text{ pts})\)

\[
y(x,t) = 0.04 \cos \left( \frac{\omega t}{2} \right) \sin \left( \frac{2\pi x}{\lambda} \right)
\]

\( \omega = 2\pi f \), \( k = \frac{2\pi}{\lambda} \)

b. What is the period, \( T \), of this wave? \((2 \text{ pts})\)

\[
T = \frac{1}{f} = \frac{1}{400} = 2.5 \times 10^{-3} \text{ s}
\]

\( T = 2.5 \times 10^{-3} \text{ s} \)

c. On the axes below, draw a sketch of the wave at the following times: \( t = 0, t = T/4, t = T/2 \). Clearly label each curve. \((2 \text{ pts})\)

d. At which value(s) of \( x \) will the transverse acceleration of the wave be largest? Compute the maximum acceleration. \((3 \text{ pts})\)

\[
\text{at } x = \frac{L}{4}, \frac{3L}{4} \Rightarrow x = \frac{0.05}{m}, \frac{0.15}{m}
\]

\[
a(t) = \frac{d^2y}{dt^2} = -0.04 \omega^2 \cos \omega t \sin \kappa x,
\]

\[
a_{\text{max}} = 0.04 \left( 2\pi f \right)^2
\]

\[
a_{\text{max}} = 2.5 \times 10^5 \text{ m/s}^2
\]
e. Standing waves can be expressed as the superposition of two traveling waves. Write down the equations for the two traveling waves that make up the standing wave in part “a”. (5 pts)

\[ y(x, t) = A \sin(kx - \omega t) + B \sin(kx + \omega t) \]

\[ k = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi}{T} \]

f. Compute the propagation speed of these traveling waves. (3 pts)

\[ v = \frac{\lambda}{T} = \lambda f = 0.20 \times 400 = 80 \text{ m/s} \]
Problem 4

Circle the correct answers

a. Suppose each polyatomic molecule in a gas has 7 degrees of freedom. What is the molar specific heat \( C_V \) for this gas assuming all degrees of freedom are accessible in equipartitioning of energy? (5pts)

i) \( C_V = 5R \)
ii) \( C_V = 5R/2 \)
iii) \( C_V = 7R/2 \) [Corrected]
iv) \( C_V = 7R \)
v) none of the above

b. A 250 g piece of lead is heated to 373 K and is then placed in a 400 g copper container holding 500 g of water. The specific heat of copper is \( C_{Cu} = 0.386 \text{ kJ/(kg K)} \). The container and the water had an initial temperature of 291 K. When thermal equilibrium is reached, the final temperature of the system is 292.15 K. If no heat has been lost from the system, what is the specific heat of the lead? [The specific heat of water is 4.180 kJ/(kg K).] (5 pts)

i) 0.110 kJ/(kg K)

ii) 0.128 kJ/(kg K) [Corrected]

iii) 0.150 kJ/(kg K)

iv) 0.0866 kJ/(kg K)

\[
C_{pb} \times 0.250 \times (292.15 - 373) \\
+ 0.386 \times 0.400 \times (292.15 - 291) \\
+ 4.18 \times 0.500 \times (292.15 - 291) = 0
\]

\[
\Rightarrow C_{pb} = 0.128 \frac{\text{kJ}}{\text{kg} \ K}
\]
c.) You are designing a post made of a single, solid column of material to prop open the mouths of wild hippos so you can clean their teeth. The post must be 0.6 m long and be able to withstand a compression force of 20,000 N without crushing. The post must also be extremely light so you can carry it easily on your trips through the jungle. You have three possible choices for materials: wood, steel, and bone. Using the data from the table on the front page, which material would make the heaviest post (and hence should be avoided)? (5 pts)

i) wood

\[ m = \rho L A \]

\[ C_{\text{max}} = 20,000 = U A \]

So, \[ m = \rho L \frac{C_{\text{max}}}{U} \]

For steel \[ m = 7860 \times 0.6 \times \frac{20,000}{400 \times 10^6} = 0.236 \text{ kg} \]

For wood \[ m = 525 \times 0.6 \times \frac{20,000}{50 \times 10^6} = 0.126 \text{ kg} \]

For bone \[ m = 1900 \times 0.6 \times \frac{20,000}{170 \times 10^6} = 0.134 \text{ kg} \]

d.) A tornado siren emits a continuous sound with power \( P = 250 \) W. The weakest intensity the human ear can hear is about \( 10^{-6} \) W/m². What is the farthest from the siren that a person can still hear the sound? (5 pts)

i) 1.5 km

ii) 2.5 km

iii) 3.5 km

iv) 4.5 km

\[ \frac{250}{4\pi R^2} = 10^{-6} \]

\[ R = \left( \frac{250 \times 10^{-6}}{4\pi} \right)^{1/2} = 4460 \text{ m} \]
An insulated cylinder whose volume is controlled by a piston of mass \( M_p \) and cross sectional area \( A \) contains gas made of atoms of mass \( m \) obeying the ideal gas law. There is no gravity nor atmospheric pressure acting on the piston, but there is a spring with spring constant \( k \) pushing against the piston in quasistatic equilibrium with the gas pressure inside the cylinder. The spring length is in its unstretched length when the volume of the cylinder is 0.

a) Suppose one measures the temperature of the gas to be \( T_0 \). How fast are “typical” gas atoms moving? Express your answer in terms of \( \{ k_B, T_0, m \} \). (5 pts)

One estimate of the typical speed of the gas molecule is the rms speed.

\[
\frac{1}{2} m v_{\text{rms}}^2 = \frac{3}{2} k_B T_0
\]

\[
v_{\text{rms}} = \sqrt{\frac{3k_B}{m}T_0}
\]

b) Suppose one heats up the gas quasi-statically from \( T_i \) to \( T_f \). What is the ratio of the final volume to the initial volume (express your answer in terms of \( T_i \) and \( T_f \))? [Hint: Recall Hooke’s law force must balance the force due to the pressure of the gas. Note that there is no gravity nor atmospheric pressure in this problem. Also, remember that Volume = \( A \times \text{height} \)] (5 pts)

ANS:

\[
\frac{T_f}{T_i} = \frac{P_f V_f}{P_i V_i}
\]

where \( V_{i,f} = h_{i,f} A \). Because of Hooke’s law, \( P_f A = k h_f \) and \( P_i A = k h_i \). Hence,

\[
\frac{T_f}{T_i} = \frac{h_f h_f}{h_i h_i}
\]

Since \( h \) and \( V \) are proportional, we find

\[
\frac{V_f}{V_i} = \sqrt{\frac{T_f}{T_i}}.
\]
c) If the initial volume is given as $V_i$, how much external work $W$ was done to the gas for the process in part b)? Assume that the answer to part b) is

$$\frac{V_f}{V_i} = c_1(\frac{T_f}{T_i})^\alpha$$

(where $V_f$ is the final volume) for particular constants $\alpha$ and $c_1$, and express your answer in terms of $\{k, T_f, T_i, V_i, A, c_1, \alpha\}$. [Hint: Volume = $A \times \text{height}$] (5 pts)

**ANS:** The work done to the gas in raising the piston by height $h$ is

$$W = -\int dV P = -\int_{h_i}^{h_f} dh (kh) = -\frac{k}{2} (h_f^2 - h_i^2)$$

$$= -\frac{k}{2A^2} (V_f^2 - V_i^2)$$

$$= -\frac{kV_i^2}{2A^2} \left(\frac{V_f}{V_i}\right)^2 - 1)$$

$$= -\frac{kV_i^2}{2A^2} (c_1(\frac{T_f}{T_i})^{2\alpha} - 1).$$

d) If the answer to part c) is $W_0$, how much heat is required for the process described in part b). (Neglect any changes in bulk mechanical energy.) Assume that the answer to part b) is

$$\frac{V_f}{V_i} = c_1(\frac{T_f}{T_i})^\alpha$$

(where $V_f$ is the final volume) for particular constants $\alpha$ and $c_1$, and express your answer in terms of $\{W_0, k, T_f, T_i, V_i, A, c_1, \alpha\}$. [Hint: The number of gas atoms can be expressed in terms of pressure, volume, and temperature.] (5 pts)

**ANS:** Energy conservation gives

$$\Delta E_{th} = W_0 + Q$$

Since the thermal energy is characterized by

$$\Delta E_{th} = \frac{3}{2} (Nk_B T_f - Nk_B T_i)$$

$$= \frac{3}{2} (P_f V_f - P_i V_i)$$

$$= \frac{3}{2} \left(\frac{kh_f}{A} V_f - \frac{kh_i}{A} V_i\right)$$

$$= \frac{3}{2} k \frac{V_f^2 - V_i^2}{A^2},$$

heat is then

$$Q = \Delta E_{th} - W_0$$

$$= \frac{3}{2} k \frac{V_f^2}{V_i^2} \left(\frac{V_f}{V_i}\right)^2 - 1) - W_0$$

$$= \frac{3}{2} k \frac{V_f^2}{V_i^2} (c_1(\frac{T_f}{T_i})^{2\alpha} - 1) - W_0.$$