Lecture 13

Goals:
• Chapter 10
  ▶ Understand the relationship between motion and energy
  ▶ Define Potential Energy in a Hooke’s Law spring
  ▶ Develop and exploit conservation of energy principle in problem solving
• Chapter 11
  ▶ Understand the relationship between force, displacement and work

Assignment:
• HW6 due Wednesday, Feb. 11
• For Thursday: Read all of Chapter 11

Energy

\[-mg \Delta y = \frac{1}{2} m (v_{y}^{2} - v_{y0}^{2})\]

\[-mg (y_f - y_i) = \frac{1}{2} m (v_{yf}^{2} - v_{yi}^{2})\]

A relationship between \(y\)-displacement and change in the \(y\)-speed

Rearranging to give initial on the left and final on the right

\[\frac{1}{2} m v_{yi}^{2} + mgy_i = \frac{1}{2} m v_{yf}^{2} + mgy_f\]

We now define \(mgy\) as the “gravitational potential energy”
Energy

- Notice that if we only consider gravity as the external force then the x and z velocities remain constant
- To \( \frac{1}{2} m v_{yi}^2 + mg y_i = \frac{1}{2} m v_{yi}^2 + mg y_f \)
- Add \( \frac{1}{2} m v_{xi}^2 + \frac{1}{2} m v_{zi}^2 \) and \( \frac{1}{2} m v_{xf}^2 + \frac{1}{2} m v_{zf}^2 \)

\[ \frac{1}{2} m v_i^2 + mg y_i = \frac{1}{2} m v_f^2 + mg y_f \]

- where \( v_i^2 = v_{xi}^2 + v_{yi}^2 + v_{zi}^2 \)

\( \frac{1}{2} m v^2 \) terms are defined to be kinetic energies (A scalar quantity of motion)

Energy

- If only “conservative” forces are present, the total energy (sum of potential, \( U \), and kinetic energies, \( K \)) of a system is conserved

For an object in a gravitational “field”

\[ \frac{1}{2} m v_{yi}^2 + mg y_i = \frac{1}{2} m v_{yi}^2 + mg y_f \]

\[ K \equiv \frac{1}{2} mv^2 \quad U \equiv mgy \]

\[ E_{mech} = K + U \]

\[ E_{mech} = K + U = \text{constant} \]

- \( K \) and \( U \) may change, but \( E_{mech} = K + U \) remains a fixed value.

\( E_{mech} \) is called “mechanical energy”
Example of a conservative system:
The simple pendulum.

- Suppose we release a mass \( m \) from rest a distance \( h_f \) above its lowest possible point.
  - What is the maximum speed of the mass and where does this happen?
  - To what height \( h_2 \) does it rise on the other side?

![Diagram of a simple pendulum]

Example: The simple pendulum.

- What is the maximum speed of the mass and where does this happen?
  - \( E = K + U = \text{constant} \) and so \( K \) is maximum when \( U \) is a minimum.
Example: The simple pendulum.

- What is the maximum speed of the mass and where does this happen?
  
  E = K + U = constant and so K is maximum when U is a minimum
  
  E = mgh, at top
  
  E = mgh₁ = ½ mv² at bottom of the swing

Example: The simple pendulum.

To what height \( h₂ \) does it rise on the other side?

E = K + U = constant and so when U is maximum again (when K = 0) it will be at its highest point.

\[ E = mgh₁ = mgh₂ \]  

or \( h₁ = h₂ \)
Example
The Loop-the-Loop ... again

- To complete the loop the loop, how high do we have to let the release the car?
- Condition for completing the loop the loop: Circular motion at the top of the loop \( a_c = \frac{v^2}{R} \)
- Use fact that \( E = U + K = \text{constant} \)

\[
U_b = mgh \\
U = mg2R \\
h ? \\
y = (\text{A}) 2R - (\text{B}) 3R - (\text{C}) 5/2 R - (\text{D}) 2^{3/2} R
\]

Recall that “g” is the source of the centripetal acceleration and \( N \) just goes to zero is the limiting case. Also recall the minimum speed at the top is

\[
v = \sqrt{gR}
\]

Example
The Loop-the-Loop ... again

- Use \( E = K + U = \text{constant} \)
- \( mgh + 0 = mg \cdot 2R + \frac{1}{2} mv^2 \)
- \( mgh = mg \cdot 2R + \frac{1}{2} mgR = 5/2 mgR \)

\[
h = 5/2 R
\]
**Example**

**Skateboard**

- What speed will the skateboarder reach halfway down the hill if there is no friction and the skateboarder starts at rest?
- Assume we can treat the skateboarder as a “point”
- Assume zero of gravitational U is at bottom of the hill

\[ m = 25 \text{ kg} \]

\[ R = 10 \text{ m} \]

\[ \theta = 30^\circ \]

\[ y = 0 \]

---

**Example**

**Skateboard**

- What speed will the skateboarder reach halfway down the hill if there is no friction and the skateboarder starts at rest?
- Assume we can treat the skateboarder as “point”
- Assume zero of gravitational U is at bottom of the hill

\[ m = 25 \text{ kg} \]

\[ R = 10 \text{ m} \]

\[ \theta = 30^\circ \]

\[ E = K + U = \text{constant} \]

\[ E_{\text{before}} = E_{\text{after}} \]

\[ 0 + m \cdot g \cdot R = \frac{1}{2} m v^2 + m g R (1 - \sin 30^\circ) \]

\[ mgR/2 = \frac{1}{2} m v^2 \]

\[ gR = v^2 \rightarrow v = (gR)^{1/2} \]

\[ v = (10 \times 10)^{1/2} = 10 \text{ m/s} \]
Potential Energy, Energy Transfer and Path

- A ball of mass $m$, initially at rest, is released and follows three different paths. All surfaces are frictionless.

1. The ball is dropped
2. The ball slides down a straight incline
3. The ball slides down a curved incline

After traveling a vertical distance $h$, how do the three speeds compare?

- (A) $1 > 2 > 3$
- (B) $3 > 2 > 1$
- (C) $3 = 2 = 1$
- (D) Can’t tell
Example
Skateboard

• What is the normal force on the skate boarder?

\[ m = 25 \text{ kg} \]
\[ R = 10 \text{ m} \]

\[ \sum F_r = ma_r = \frac{m v^2}{R} \]
\[ = N - mg \cos 60^\circ \]

\[ N = 25 \times 100 \div 10 + 25 \times 10 \times (0.87) \]
\[ N = 250 + 220 = 470 \text{ Newtons} \]
Elastic vs. Inelastic Collisions

- A collision is said to be *elastic* when energy as well as momentum is conserved before and after the collision.
  \[ K_{\text{before}} = K_{\text{after}} \]
  - Carts colliding with a perfect spring, billiard balls, etc.

---

Elastic vs. Inelastic Collisions

- A collision is said to be *inelastic* when energy is not conserved before and after the collision, but momentum is conserved.
  \[ K_{\text{before}} \neq K_{\text{after}} \]
  - Car crashes, collisions where objects stick together, etc.
Inelastic collision in 1-D: Example 1

- A block of mass $M$ is initially at rest on a frictionless horizontal surface. A bullet of mass $m$ is fired at the block with a muzzle velocity (speed) $v$. The bullet lodges in the block, and the block ends up with a speed $V$.

- What is the initial energy of the system?
- What is the final energy of the system?
- Is energy conserved?

\[ \begin{array}{ccc}
& V & \\
before & & after \\
\end{array} \]

Physics 207: Lecture 13, Pg 19

Inelastic collision in 1-D: Example 1

What is the momentum of the bullet with speed $v$? $m\vec{v}$

- What is the initial energy of the system? $\frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m v^2$
- What is the final energy of the system? $\frac{1}{2} (m + M) V^2$
- Is momentum conserved (yes)? $m v + M 0 = (m + M) V$
- Is energy conserved? Examine $E_{\text{before}} - E_{\text{after}}$

\[ \frac{1}{2} m v^2 - \frac{1}{2} [(m + M) V]^2 = \frac{1}{2} m v^2 - \frac{1}{2} (m v) \frac{m}{m + M} v = \frac{1}{2} m v^2 \left( 1 - \frac{m}{m + M} \right) \]

\[ \begin{array}{ccc}
& V & \\
before & & after \\
\end{array} \]
Example – Fully Elastic Collision

- Suppose I have 2 identical bumper cars.
- One is motionless and the other is approaching it with velocity $v_1$. If they collide elastically, what is the final velocity of each car?
  
  Identical means $m_1 = m_2 = m$
  Initially $v_{\text{Green}} = v_1$ and $v_{\text{Red}} = 0$

\[
\begin{align*}
\text{COM} & \rightarrow \quad m v_1 + 0 = m v_{1f} + m v_{2f} \quad \Rightarrow \quad v_1 = v_{1f} + v_{2f} \\
\text{COE} & \rightarrow \quad \frac{1}{2} m v_1^2 = \frac{1}{2} m v_{1f}^2 + \frac{1}{2} m v_{2f}^2 \quad \Rightarrow \quad v_1^2 = v_{1f}^2 + v_{2f}^2 \\
\end{align*}
\]

- $v_{1f}^2 = (v_{1f} + v_{2f})^2 = v_{1f}^2 + 2v_{1f}v_{2f} + v_{2f}^2 \Rightarrow 2v_{1f}v_{2f} = 0$
- Soln 1: $v_{1f} = 0$ and $v_{2f} = v_1$  
  Soln 2: $v_{2f} = 0$ and $v_{1f} = v_1$

Variable force devices: Hooke’s Law Springs

- Springs are everywhere, (probe microscopes, DNA, an effective interaction between atoms)

\[
F_s = -k \Delta s
\]

$\Delta s$ is the amount the spring is stretched or compressed from it resting position.
**Exercise 2**
**Hooke's Law**

What is the spring constant “k”?

\[ \Sigma F = 0 = F_s - mg = k \Delta s - mg \]

Use \( k = \frac{mg}{\Delta s} = \frac{500 \text{ N}}{1.0 \text{ m}} \)

(A) 50 N/m  (B) 100 N/m  (C) 400 N/m  (D) 500 N/m
F-s relation for a foot arch:

<table>
<thead>
<tr>
<th>Force (N)</th>
<th>Displacement (mm)</th>
</tr>
</thead>
</table>

Force vs. Energy for a Hooke’s Law spring

- $F = -k(x - x_{\text{equilibrium}})$
- $F = ma = m \frac{dv}{dt}$
  - $= m \left(\frac{dv}{dx} \frac{dx}{dt}\right)$
  - $= mv \frac{dv}{dx}$
- So $-k(x - x_{\text{equilibrium}}) \frac{du}{dx} = mv \frac{dv}{du}$
- Let $u = x - x_{\text{eq.}} \Rightarrow du = dx$ \[ \int_{x_i}^{x_f} -k \, du = \int_{v_i}^{v_f} mv \, dv \]
- $-\frac{1}{2} ku^2 \bigg|_{x_i}^{x_f} = \frac{1}{2} mv^2 \bigg|_{v_i}^{v_f}$
- $-\frac{1}{2} kx_f^2 + \frac{1}{2} kx_i^2 = \frac{1}{2} mv_f^2 - \frac{1}{2} mv_i^2$

\[ \frac{1}{2} kx_i^2 + \frac{1}{2} mv_i^2 = \frac{1}{2} kx_f^2 + \frac{1}{2} mv_f^2 \]
Energy for a Hooke’s Law spring

\[
\frac{1}{2} k x_i^2 + \frac{1}{2} m v_i^2 = \frac{1}{2} k x_f^2 + \frac{1}{2} m v_f^2
\]

- Associate \( \frac{1}{2} k x^2 \) with the “potential energy” of the spring

\[
U_{si} + K_i = U_{sf} + K_f
\]

- Hooke’s Law springs are conservative so the mechanical energy is constant

Energy diagrams

- In general:

Ball falling

\[
E_{\text{mech}}
\]

Energy

\[
y
\]

Spring/Mass system

\[
E_{\text{mech}}
\]

Energy

\[
s
\]
Energy diagrams

Spring/Mass/Gravity system

- Energy diagrams
  - Spring alone
  - Spring & gravity

Force

Equilibrium

- Example
  - Spring: $F_x = 0 \Rightarrow \frac{dU}{dx} = 0$ for $x=x_{eq}$
    The spring is in equilibrium position

- In general: $\frac{dU}{dx} = 0 \Rightarrow$ for ANY function establishes equilibrium

stable equilibrium

unstable equilibrium
Comment on Energy Conservation

- We have seen that the total kinetic energy of a system undergoing an inelastic collision is not conserved.
  - Mechanical energy is lost:
    - Heat (friction)
    - Bending of metal and deformation

- Kinetic energy is not conserved by these non-conservative forces occurring during the collision!

- Momentum along a specific direction is conserved when there are no external forces acting in this direction.
  - In general, easier to satisfy conservation of momentum than energy conservation.

Lecture 13

Assignment:
- HW6 due Wednesday 2/11
- For Monday: Read all of chapter 11