Relativity

Relativistic Mechanics

Stars observed orbiting a black hole.
Relativistic Mechanics

Relativistic space and time requires generalization of Newtonian mechanics

\[ p = mv \Rightarrow p = \gamma mv \]
\[ E = \frac{1}{2}mv^2 \Rightarrow E = \gamma mc^2 \]
\[ ma = F \Rightarrow \frac{dp}{dt} = F \]

Today we motivate and apply relativistic mechanics.
Momentum conservation

Consider in the cm (S) frame a collision of two equal masses which stick together:

1. Before
   - Mass 1: $m$, velocity $v$
   - Mass 2: $m$, velocity $v$

2. After
   - Combined mass: $2m$
   - Velocity: $V = 0$

Momentum is conserved according to S

\[ p_{\text{before}} = m v + m(-v) = 0 \]
\[ p_{\text{after}} = 0 \]
P=mv not conserved

Consider the same collision in the rest frame of mass 1. The relative velocity of the two frames is \( v \).

Use \( v_2 = -v \) and rel. velocity addition to find \( p=mv \):
\[
p'_{\text{before}} = m v_2' = m(v_2 - v)/[1 - v_2 v/c^2] = -2mv/ [1 + v^2/c^2]
\]

Use the relative velocity to find:
\[
p'_{\text{after}} = -2mv \text{ which is not equal to } p'_{\text{before}}
\]
Relativistic momentum

\[ p = \gamma mv = \frac{1}{\sqrt{1-(v/c)^2}} mv \]

For \( v << c \), neglect the denominator and \( p = mv \) as before.

Note that as the velocity approaches the maximum value (light speed), the momentum continues to increase without bound.
Relativistic energy

\[ E = \gamma mc^2 = \frac{1}{\sqrt{1 - (v/c)^2}} mc^2 \]

As the \( v \to c \), the energy increases without bound and \( E = pc \). The non-relativistic limit is

\[ \frac{1}{\sqrt{1 - (v/c)^2}} \approx 1 + \frac{1}{2} (v/c)^2 + \ldots \]

\[ E = mc^2 + \frac{1}{2} mv^2 + \ldots = mc^2 + KE_{\text{nonrelativistic}} + \ldots \]
Rest energy

\[ E = mc^2 \]

Einstein’s famous formula describes the energy of a mass \( m \) in its rest frame.
Some values

Rest energy of 1 kg of matter:
$E = 1 \text{ kg} \ (3\times10^8 \text{ m/s})^2 = 9\times10^{16} \text{ Joules}$

Compare to gravitational energy $U=Mgh$.
The rest energy of 1 kg can lift a mass $M=9\times10^{15} \text{ kg}$
a vertical height $h = \frac{E}{Mg} = \frac{9\times10^{16}}{(9\times10^{15}\times9.8)} = 1 \text{ m}$

Rest energies of some subatomic particles:

$$m_e c^2 = 0.511 \text{ MeV}$$
$$m_p c^2 = 938 \text{ MeV}$$
Relationships between $E$ and $p$

\[
E^2 - c^2 p^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 c^2 \\
= m^2 c^4 \gamma^2 (1 - (v/c)^2) = m^2 c^4 \\
E = \sqrt{(pc)^2 + (mc^2)^2}
\]

Also

\[
E = \gamma mc^2, \quad p = \gamma mv \Rightarrow v = p[c^2]/E
\]
“Set c=1”

If you measure energy in MeV and momentum in MeV/c and velocity in units of light speed, you can drop the clutter of “c”s.

A proton has rest energy (mass) 0.938 GeV and momentum 1 GeV/c. What is its total energy in GeV and speed relative to light speed?

\[
E = \sqrt{p^2 + m^2} \approx \sqrt{1 + 1} = p\sqrt{2} \\
v'/c = p/E = 1/\sqrt{2}
\]
Relativistic force law

\[ \frac{dp}{dt} = \frac{d}{dt} (\gamma mv) = F \]

In the absence of forces, momentum is constant and particles travel in straight lines with constant speed.

A force increases momentum but acceleration is not proportional to force unless \( v \ll c \).

If \( v \ll c \), \( p = mv \) and we recover Newton’s 2nd Law of motion: \( F = ma \).
Constant force motion

\[ \frac{dp}{dt} = F \Rightarrow \]
\[ p = \frac{mv}{\sqrt{1-(v/c)^2}} = Ft \Rightarrow \]
\[ v = c\left(\frac{Ft}{m}\right)/\sqrt{1 + \left(\frac{Ft}{m}\right)^2} \]

A force increases momentum without limit but the velocity does not exceed c.
Force and energy

\[ E = \sqrt{p^2 + m^2} \Rightarrow dE = \frac{pd\nu}{E} \Rightarrow \]
\[ \frac{dE}{dt} = \nu \frac{dp}{dt} = \nu F = \frac{dx}{dt} F \Rightarrow \]
\[ \Delta E = \int dE = \int dx F \]

The change in the relativistic energy is the work done by the force, as in non-relativistic mechanics.
Example

An electron starts at rest and is accelerated through a potential difference $V = 1$ MV. What is its relativistic energy and momentum and speed?
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An electron starts at rest and is accelerated through a potential difference $V= 1$ MV. What is its relativistic energy and momentum and speed?

The change in energy is the work $W=qV = 1$ MeV. The electron rest energy is $m[c^2 ] = 0.511$ MeV so the total energy is $E = m[c^2] + W = 1.511$ MeV.

The momentum (in MeV/c) is
$$p = [E^2 - (mc^2)^2]^{1/2} = 1.42$$

The speed is $v/c=p[c]/E = 1.42/1.51=0.94$
Motion in a uniform constant magnetic field

Magnetic force does no work so $E$ is constant

$$\frac{dp}{dt} = F = qE + qv \times B = qv \times B$$

$$F \cdot v = 0 \Rightarrow E = \text{constant}$$

Constant $E$ means constant $v$ so

$$\frac{dp}{dt} = \frac{d[\gamma m v]}{dt} = \gamma m \frac{dv}{dt} = qv \times B$$

This is the non-relativistic form with $m \Rightarrow \gamma m$
Magnetic radius example

What is the radius of the circular orbit of a 1000 GeV proton in a uniform magnetic field of 4 tesla?
Magnetic radius example

What is the radius of the circular orbit of a 1000 GeV proton in a uniform magnetic field of 4 tesla?

The non-relativistic radius is $R = \frac{mv}{qB}$ where $q = e$. Replace $m$ with gamma times $m$.

The proton mass is 938 MeV or about 1 GeV so the gamma factor is about $\frac{1000}{.938} = 1066$ and $v \sim c$.

$$R = \frac{(1066)(1.67e^{-27} \text{ kg})(3e8 \text{ m/s})}{(1.6e-19 \text{ C})(4 \text{ T})} = 834 \text{ m}$$
Energy/momentum conserved

In non-relativistic mechanics, the total momentum of a closed system is conserved. Energy but not kinetic energy is conserved. Kinetic energy may be converted from or to potential or thermal energy.

Relativistic energy and momentum of a closed system are always conserved.

Inelasticity is reflected in changes in the inertial mass.
Revisit our example collision

Consider in the cm (S) frame a collision of two equal masses which stick together:

Momentum is conserved according to S

\[ p_{\text{before}} = \gamma_v m v + \gamma_{-v} m (-v) = 0 \]
\[ p_{\text{after}} = 0 + 0 = 0 \]

KE is not conserved.
Inelastic collision example

Relativistic momentum in our example collision. (See ex. 2.6 in text)

From velocity addition $v_2' = -2mv/ [1+v^2/c^2]$ so $p'_{\text{before}} = m v_2' / [1-(v_2'/c)^2]^{1/2} = -2mv/ [1-v^2/c^2]$.

The speed of the stuck masses is $V = -v$ so $p'_{\text{after}} = -M v / [1-(v/c)^2]^{1/2}$.

Energy conservation in the cm tells us $E_{\text{before}} = E_1 + E_2 = 2 mc^2 / [1-(v/c)^2]^{1/2} = E_{\text{after}} = Mc^2$.

so $M = 2m [1-(v/c)^2]^{1/2}$ so and $p'_{\text{before}} = p'_{\text{after}}$

Energy and momentum are both conserved.
Example of a 2-body decay, pion to muon plus neutrino.
Pion decay

In any frame of reference

\[ E_\pi = E_\mu + E_\nu; \vec{p}_\pi = \vec{p}_\mu + \vec{p}_\nu \]

In the rest frame of the pion (c=1)

\[ m_\pi = E_\mu + E_\nu; 0 = p_\mu + p_\nu \]

\[ E_\mu = \sqrt{p_{\mu}^2 + m_{\mu}^2}; E_\nu = \sqrt{p_{\nu}^2 + m_{\nu}^2} \]

Two equations in two unknowns.
Pion decay continued

Use momentum conservation

\[ p_\mu = -p_\nu \equiv p \]

Then energy conservation reads

\[ m_\pi = \sqrt{p^2 + m_\mu^2} + \sqrt{p^2 + m_\nu^2} \]

Solve this quadratic equation for \( p \). Then we can compute the energies.

\[ m_\pi = 140 \text{ MeV}; \ m_\mu = 105 \text{ MeV}; \ m_\nu = 0 \]

In this particular case, one particle (neutrino) is massless.
Pion decay continued

The decay product energy and momentum may be expressed in terms of the masses

\[ E_\nu = \frac{m_\pi^2 - m_\mu^2 + m_\nu^2}{2m_\pi} = 30 \text{ MeV} \]

A two body decay produces a unique momentum. In a three body decay (Mercedes-Benz), the released energy is shared in a way that depends on the angles between the three particles.
Principle of equivalence

a) Standing on the ground you feel your weight mg.
b) Accelerating upwards with a=g feels the same.
c) Relative to an accelerating frame, a horizontal light beam appears to fall.

d) In a gravitational field, does light fall?
Inertial and gravitational mass

a) Standing on the ground you feel your weight mg.
b) Accelerating upwards with $a=g$ feels the same.

In Newtonian mechanics, the equivalence of a gravitational field to an acceleration of reference frame follows from the equality of inertial and gravitational mass:

\[
F = m_I a \\
F_{\text{gravity}} = m_G g \\
m_I = m_G => a = g \text{ for ANY object.}
\]
Deflection of starlight observed

Light does fall in a gravitational field, but not much usually.
An exercise 2-25

You will calculate the deflection assuming light is composed of nearly massless particles.

The answer is of the order of what is observed, but wrong.
Gravitational lensing

Astronomy Picture of the Day

Discover the cosmos! Each day we feature a different image or photograph of our fascinating universe, along with a brief explanation written by a professional astronomer.

July 11, 1995

Microlensing of the Einstein Cross
Picture Credit: Geraint Lewis and Michael Irwin, William Hershel Telescope

Explanation: The famous "Einstein Cross" is a case where a single object is seen four times. Here a very distant QSO happened to be placed right behind a massive galaxy. The gravitational effect of the galaxy on the distant QSO was similar to the lens effect of a drinking glass on a distant street light - it created multiple images. But stars in the foreground galaxy have been found to act as gravitational lenses here too! These stars make the images change brightness relative to each other. These brightness changes are visible on these two photographs of the Einstein Cross, taken about 3 years apart.

For more information about this picture see the home page of the IAU Symposium 173 Astrophysical Applications of Gravitational Lensing in Melbourne, Australia 9-14 July, 1995.
General Relativity

Principle of general relativity:

A free falling frame is physically equivalent to an inertial frame. Gravity is physically equivalent to acceleration.

If you are in free falling elevator, relative to you, free particles move in straight lines with constant velocity. For example, particles at rest stay at rest.

More generally, all physics works locally as if you were free of a gravitational field in a frame with constant velocity relative to the “fixed” stars.
Consequences

Like masses, light travels horizontally relative to a free falling elevator, and must fall with the elevator relative to a ground observer, so light MUST fall in a gravitational field.
Connection with geometry

Our ideas about geometry comes from vision - light signals from objects. We use light beams in survey work to define straight lines.

Light beams are affected by gravity. Geometry as defined operationally, as a physical science as opposed to a mathematical theory, depends on gravity.

Back to square one. Pythagorean geometry is only approximately correct.
Gravitational frequency shift

Fire light up in a free falling elevator. Relative to the elevator, light moves freely from bottom to top. Relative to earth, by the time $dt = L/c$ when it arrives at the top, the elevator speed has increased by $dv = gd dt$, and to an elevator observer the light is Doppler shifted to higher frequency by $df = (v/c)f$.

Resolution: an atom (clock) at the bottom of the elevator runs more slowly that a similar atom higher up - time depends on gravitational field!
Curved space time

Einstein concluded that gravity is associated with distortion in space and time and set about to discover equations relating curvature to matter/energy density.
Sketch of the theory

Distances in non-Cartesian space are described by a metric tensor field $g_{ij}(x)$.

\[
ds^2 = dx^2 + dy^2 + dz^2 \Rightarrow \\
ds^2 = \sum_{i,j=1}^{4} g_{ij}(x) dx^i dx^j
\]
Connection with matter

Curvature away from flat in any local plan may be expressed in terms of the metric tensor and (Einstein postulated) is proportional to energy density

\[ C = \frac{8\pi G_{\text{Newton}}}{c^2} \rho E \]
Estimate of curvature of space

Guess form of curvature in x-y plane near the Earth

\[ C = \frac{MGc^a}{r^b} \]

If \( C \) has dimensions of \((1/R_x)(1/R_y)\) we need \( a=-2, b=3 \).

\[
\frac{C_{\text{Earth}}}{1/R_{\text{Earth}}^2} = \frac{M_{\text{Earth}} G}{c^2 R_{\text{Earth}}} = g R_{\text{Earth}} / c^2 \sim 10^{-9}
\]

The distortion of space is small near the Earth’s surface where gravity is relatively weak.
Tests of theory

Predict deflection of light correctly.
Planetary motion

The effective gravitational force outside a mass such as the sun is not $GMm/r^2$ and hence the orbits of the planets are not ellipses.

The orbital ellipses precess - the axes slowly rotate.

The amounts are exactly as predicted by Einstein’s general theory.
Einstein’s theory of motion

Newtonian theory:
  Mass generates force field
  \( F=ma \) predicts motion in a given field

Einstein’s theory:
  Mass generates curvature in space and time
  Mass move along geodesics (minimal length paths)
Black holes

Light can fall. Can it go into orbit? Yes. Can it be trapped? Yes!
Black holes

The kinetic energy for mass m to escape a the surface at radius R of mass M defines the escape speed:

\[ \frac{1}{2} m v^2 = m M G / R \Rightarrow \]
\[ v = \sqrt{2 M G / R} \]

If \( v = c \), escape is impossible. That is the case if the mass all is inside the Schwarzschild radius

\[ R = \frac{2 G M}{c^2} \]
The road to black hole formation

Collapsing normal stars produce white dwarf stars, neutron stars, black holes depending on their mass.
Gravitation radiation

Like the electromagnetism, general relativity predicts accelerated masses radiate gravity waves that travel at light speed, disturbances in space-time.
Radiation by binary stars

The theory permits calculation of the energy loss due to radiation by co-orbiting binary stars.
Confirmation of gravitational radiation

Pulsars are binaries which appear to pulse once per orbit. The energy loss due to gravitational radiation can be seen in the slow change of the orbital period.

Present direct searches

Very long interferometers are used to search directly for gravity radiation from nearby collapsing single and binary stellar systems.
Our Black Hole
Inference from the orbits

The star swings by the hole at a minimum distance $b$ of 17 light hours (120 A.U. or close to three times the distance to Pluto) at speed $v = 5000$ km/s. The orbit is highly elliptical - at the minimum distance, the focus of attraction is $1 - \epsilon = 0.13$ times half the length $a$ of the major axis of the ellipse. The period is 15.2 years. Newtonian mechanics gives the relationship $v^2 = GM(2/b - 1/a)$ and Kepler's Third Law $T^2 = (2\pi)^2 a^3/MG$ from which the mass $M$ is found to be $3.7 \pm 1.5$ million solar masses all within a radius less than $1.8e13$ m. Combining observations of several stars improves the mass estimate to $2.6 \pm 0.2$ million solar masses.

http://www.mpe.mpg.de/ir/GC/intro.html