Chapter 37. Relativity

You can measure lengths with a ruler or meter stick. You can time events with a stopwatch. Einstein’s special theory of relativity questions deeply held assumptions about the nature of space and time.

Chapter Goal: To understand how Einstein’s theory of relativity changes our concepts of space and time.
Chapter 37. Relativity

Topics:
• Relativity: What’s It All About?
• Galilean Relativity
• Einstein’s Principle of Relativity
• Events and Measurements
• The Relativity of Simultaneity
• Time Dilation
• Length Contraction
• The Lorentz Transformations
• Relativistic Momentum
• Relativistic Energy
Physical observations are referred to a frame of coordinates and synchronized clocks called a ‘frame of reference.’

An **inertial frame** is one in uniform motion relative to the stars and is defined by the condition that free particles (of matter and light) move with uniform velocity.
The Galilean Transformations

Consider two inertial frames $S$ and $S'$. The coordinate axes in $S$ are $x, y, z$ and those in $S'$ are $x', y', z'$. Reference frame $S'$ moves with velocity $v$ relative to $S$ along the $x$-axis. Equivalently, $S$ moves with velocity $-v$ relative to $S'$.

The *Galilean transformations of position* are:

\[
\begin{align*}
x &= x' + vt \\
y &= y' \\
z &= z'
\end{align*}
\]

or

\[
\begin{align*}
x' &= x - vt \\
y' &= y \\
z' &= z
\end{align*}
\]

Implicitly, the time coordinate is assumed to be the same in both frames.

The *Galilean transformations of velocity* are:

\[
\begin{align*}
\mathbf{u}_x &= \mathbf{u}_x' + v \\
\mathbf{u}_y &= \mathbf{u}_y' \\
\mathbf{u}_z &= \mathbf{u}_z'
\end{align*}
\]

or

\[
\begin{align*}
\mathbf{u}_x' &= \mathbf{u}_x - v \\
\mathbf{u}_y' &= \mathbf{u}_y \\
\mathbf{u}_z' &= \mathbf{u}_z
\end{align*}
\]
Einstein’s Principle of Relativity

**Principle of relativity** All the laws of physics are the same in all inertial reference frames.

- Maxwell’s equations are true in all inertial reference frames.
- Maxwell’s equations predict that electromagnetic waves, including light, travel at speed \( c = 3.00 \times 10^8 \) m/s.
- Therefore, light travels at speed \( c \) in all inertial reference frames.

*Every* experiment has found that light travels at \( 3.00 \times 10^8 \) m/s in every inertial reference frame, regardless of its velocity.
FIGURE 37.9 Light travels at speed \( c \) in all inertial reference frames, regardless of how the reference frames are moving with respect to the light source.

This light wave leaves Amy at speed \( c \) relative to Amy. It approaches Cathy at speed \( c \) relative to Cathy.

This light wave leaves Bill at speed \( c \) relative to Bill. It approaches Cathy at speed \( c \) relative to Cathy.
Consider light bouncing between mirrors transverse to a motion of $S'$ relative to $S$. Relative to $S$, the light moves a further distance between ticks yet at the same speed so the clock rate appears decreased compared to the same clock at rest.
The time interval between two events that occur at the same position is called the **proper time** $\Delta \tau$. In an inertial reference frame moving with velocity $v = \beta c$ relative to the proper time frame, the time interval between the two events is

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - \beta^2}} \geq \Delta \tau \quad \text{(time dilation)}$$

The “stretching out” of the time interval is called **time dilation**.
Time Dilation Evidence

The decays of relativistic particle provided early evidence for time dilation. A heavy electron called a muon decays to an electron plus neutrinos in a proper time of about 2 microseconds. A relativistic muon in motion decays more slowly and travels farther.

Time dilation is verified directly by flying atomic clocks on extended airline trips. The slowing down of the moving clock is a small effect because $v/c \ll 1$. 
EXAMPLE 37.5 From the sun to Saturn

QUESTIONS:

EXAMPLE 37.5 From the sun to Saturn
Saturn is $1.43 \times 10^{12}$ m from the sun. A rocket travels along a line from the sun to Saturn at a constant speed of $0.9c$ relative to the solar system. How long does the journey take as measured by an experimenter on earth? As measured by an astronaut on the rocket?
EXAMPLE 37.5 From the sun to Saturn

Take $S$ to be the frame of the solar system; take $S'$ to be the frame of the rocket.
EXAMPLE 37.5 From the sun to Saturn

**SOLVE** The time interval measured in the solar system reference frame, which includes the earth, is simply

\[
\Delta t = \frac{\Delta x}{v} = \frac{1.43 \times 10^{12} \text{ m}}{0.9 \times (3.00 \times 10^8 \text{ m/s})} = 5300 \text{ s}
\]

Relativity hasn’t abandoned the basic definition \( v = \frac{\Delta x}{\Delta t} \), although we do have to be sure that \( \Delta x \) and \( \Delta t \) are measured in just one reference frame and refer to the same two events.
EXAMPLE 37.5 From the sun to Saturn

How are things in the rocket’s reference frame? The two events occur at the *same position* in $S'$ and can be measured by one clock, the clock at the origin. Thus the time measured by the astronauts is the *proper time* $\Delta \tau$ between the two events. We can use Equation 37.9 with $\beta = 0.9$ to find

$$\Delta \tau = \sqrt{1 - \beta^2} \Delta t = \sqrt{1 - 0.9^2} (5300 \text{ s}) = 2310 \text{ s}$$
EXAMPLE 37.5 From the sun to Saturn

**ASSESS** The time interval measured between these two events by the astronauts is less than half the time interval measured by experimenters on earth. The difference has nothing to do with when earthbound astronomers *see* the rocket pass the sun and Saturn. $\Delta t$ is the time interval from when the rocket actually passes the sun, as measured by a clock at the sun, until it actually passes Saturn, as measured by a synchronized clock at Saturn. The interval between *seeing* the events from earth, which would have to allow for light travel times, would be something other than 5300 s. $\Delta t$ and $\Delta \tau$ are different because *time is different* in two reference frames moving relative to each other.
Length Contraction

In the rocket frame, Saturn moves toward the rocket at speed $v$ (velocity $-v$) and arrives after a time interval shorter than the interval in the solar system frame. The distance must be less (contracted) by the same factor as the time interval is dilated.
The distance $L$ between two objects, or two points on one object, measured in the reference frame $S$ in which the objects are at rest is called the **proper length** $\ell$. The distance $L'$ in a reference frame $S'$ is

$$L' = \sqrt{1 - \beta^2} \ell \leq \ell$$

**NOTE:** Length contraction does not tell us how an object would *look*. The visual appearance of an object is determined by light waves that arrive simultaneously at the eye. Length and length contraction are concerned only with the *actual* length of the object at one instant of time.
EXAMPLE 37.6 The distance from the sun to Saturn

QUESTION:

In Example 37.5 a rocket traveled along a line from the sun to Saturn at a constant speed of $0.9c$ relative to the solar system. The Saturn-to-sun distance was given as $1.43 \times 10^{12}$ m. What is the distance between the sun and Saturn in the rocket’s reference frame?
EXAMPLE 37.6 The distance from the sun to Saturn

**MODEL** Saturn and the sun are, at least approximately, at rest in the solar system reference frame $S$. Thus the given distance is the proper length $\ell$. 
EXAMPLE 37.6 The distance from the sun to Saturn

**SOLVE** We can use Equation 37.15, to find the distance in the rocket’s frame \( S' \):

\[
L' = \sqrt{1 - \beta^2} \ell = \sqrt{1 - 0.9^2} (1.43 \times 10^{12} \text{ m})
\]

\[
= 0.62 \times 10^{12} \text{ m}
\]

**ASSESS** The sun-to-Saturn distance measured by the astronauts is less than half the distance measured by experimenters on earth. \( L' \) and \( \ell \) are different because *space is different* in two reference frames moving relative to each other.
The Lorentz Transformations

Consider two reference frames S and S'. An event occurs at coordinates $x, y, z, t$ as measured in S, and the same event occurs at $x', y', z', t'$ as measured in S'. Reference frame S' moves with velocity $v$ relative to S, along the $x$-axis.

The **Lorentz transformations** for the coordinates of one event are:

\[
\begin{align*}
    x' &= \gamma(x - vt) & x &= \gamma(x' + vt') \\
    y' &= y & y &= y' \\
    z' &= z & z &= z' \\
    t' &= \gamma(t - vx/c^2) & t &= \gamma(t' + vx'/c^2)
\end{align*}
\]

\[
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}
\]
The Lorentz Velocity Transformations

Consider two reference frames S and S'. An object moves at velocity $u$ along the $x$-axis as measured in S, and at velocity $u'$ as measured in S'. Reference frame S' moves with velocity $v$ relative to S, also along the $x$-axis.

The Lorentz velocity transformations are:

$$u' = \frac{u - v}{1 - uv/c^2} \quad \text{and} \quad u = \frac{u' + v}{1 + u'v/c^2}$$

**NOTE:** It is important to distinguish carefully between $v$, which is the relative velocity between two reference frames, and $u$ and $u'$ which are the velocities of an object as measured in the two different reference frames.
The Lorentz velocity transformations follow from the Lorentz transformation:

\[
\begin{align*}
\frac{dx'}{dt'} &= \gamma \left( dt - v \frac{dx}{c^2} \right) = \frac{dx/dt - v}{1 - v \frac{dx}{dt} / c^2} \\
\end{align*}
\]

\[
\begin{align*}
u' &= \frac{u - v}{1 - vu/c^2}
\end{align*}
\]

Note: When the object moves at light speed \( u = c \) and

\[
\begin{align*}
u' &= \frac{c - v}{1 - vc/c^2} = c
\end{align*}
\]
EXAMPLE 37.10 A really fast bullet

QUESTION:

A rocket flies past the earth at 0.90c. As it goes by, the rocket fires a bullet in the forward direction at 0.95c with respect to the rocket. What is the bullet’s speed with respect to the earth?
EXAMPLE 37.10 A really fast bullet

**MODEL**  The rocket and the earth are inertial reference frames. Let the earth be frame S and the rocket be frame S'. The velocity of frame S' relative to frame S is \( v = 0.90c \). The bullet’s velocity in frame S’ is \( u' = 0.95c \).

**SOLVE**  We can use the Lorentz velocity transformation to find

\[
 u = \frac{u' + v}{1 + u'v/c^2} = \frac{0.95c + 0.90c}{1 + (0.95c)(0.90c)/c^2} = 0.997c
\]

**NOTE**  Many relativistic calculations are much easier when velocities are specified as a fraction of \( c \).
EXAMPLE 37.10 A really fast bullet

**ASSESS** In Newtonian mechanics, the Galilean transformation of velocity would give \( u = 1.85c \). Now, despite the very high speed of the rocket and of the bullet with respect to the rocket, the bullet’s speed with respect to the earth remains less than \( c \). This is yet more evidence that objects cannot exceed the speed of light.
Relativistic Momentum

The momentum of a particle moving at speed $u$ is

$$ p = \gamma_p mu $$

where the subscript $p$ indicates that this is $\gamma$ for a particle, not for a reference frame.

- If $u \ll c$, the momentum approaches the Newtonian value of $p = mu$. As $u$ approaches $c$, however, $p$ approaches infinity.
- For this reason, a force cannot accelerate a particle to a speed higher than $c$, because the particle’s momentum becomes infinitely large as the speed approaches $c$. 
Relativistic Energy

The total energy $E$ of a particle is

$$E = \gamma_p mc^2 = E_0 + K = \text{rest energy} + \text{kinetic energy}$$

This total energy consists of a rest energy

$$E_0 = mc^2$$

and a relativistic expression for the kinetic energy

$$K = (\gamma_p - 1)mc^2 = (\gamma_p - 1)E_0$$

This expression for the kinetic energy is very nearly $\frac{1}{2}mu^2$ when $u << c$. 
Conservation of Energy

The creation and annihilation of particles with mass, processes strictly forbidden in Newtonian mechanics, are vivid proof that neither mass nor the Newtonian definition of energy is conserved. Even so, the total energy—the kinetic energy and the energy equivalent of mass—remains a conserved quantity.

Law of conservation of total energy  The energy $E = \sum E_i$ of an isolated system is conserved, where $E_i = (\gamma_p) i m_i c^2$ is the total energy of particle $i$.

Mass and energy are not the same thing, but they are equivalent in the sense that mass can be transformed into energy and energy can be transformed into mass as long as the total energy is conserved.
**FIGURE 37.41** In nuclear fission, the energy equivalent of lost mass is converted into kinetic energy.

The mass of the reactants is 0.185 u more than the mass of the products.

0.185 u of mass has been converted into kinetic energy.
Energy in mass

Consider the energy equivalent of 1 kilogram.

\[ E = mc^2 = (1 \, kg)(3 \times 10^8 \, m/s)^2 = 9 \times 10^{16} \, joule \]
EXAMPLE 37.11 Momentum of a subatomic particle

**QUESTION:**

Electrons in a particle accelerator reach a speed of $0.999c$ relative to the laboratory. One collision of an electron with a target produces a muon that moves forward with a speed of $0.95c$ relative to the laboratory. The muon mass is $1.90 \times 10^{-28}$ kg. What is the muon’s momentum in the laboratory frame and in the frame of the electron beam?
EXAMPLE 37.11 Momentum of a subatomic particle

**MODEL** Let the laboratory be reference frame S. The reference frame $S'$ of the electron beam (i.e., a reference frame in which the electrons are at rest) moves in the direction of the electrons at $v = 0.999c$. The muon velocity in frame S is $u = 0.95c$. 
EXAMPLE 37.11 Momentum of a subatomic particle

\textbf{SOLVE} \ \gamma_p \text{ for the muon in the laboratory reference frame is}

\[ \gamma_p = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - 0.95^2}} = 3.20 \]

Thus the muon’s momentum in the laboratory is

\[ p = \gamma_p m_u = (3.20)(1.90 \times 10^{-28} \text{ kg})(0.95 \times 3.00 \times 10^8 \text{ m/s}) \]

\[ = 1.73 \times 10^{-19} \text{ kg m/s} \]

The momentum is a factor of 3.2 larger than the Newtonian momentum \( m_u \). To find the momentum in the electron-beam reference frame, we must first use the velocity transformation equation to find the muon’s velocity in frame \( S' \):

\[ u' = \frac{u - v}{1 - uv/c^2} = \frac{0.95c - 0.999c}{1 - (0.95c)(0.999c)/c^2} = -0.962c \]
EXAMPLE 37.11 Momentum of a subatomic particle

In the laboratory frame, the faster electrons are overtaking the slower muon. Hence the muon’s velocity in the electron-beam frame is negative. $\gamma'_p$ for the muon in frame S’ is

$$\gamma'_p = \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.962^2}} = 3.66$$

The muon’s momentum in the electron-beam reference frame is

$$p' = \gamma'_p m u'$$

$$= (3.66)(1.90 \times 10^{-28} \text{ kg})(-0.962 \times 3.00 \times 10^8 \text{ m/s})$$

$$= -2.01 \times 10^{-19} \text{ kg m/s}$$
EXAMPLE 37.11 Momentum of a subatomic particle

ASSESS From the laboratory perspective, the muon moves only slightly slower than the electron beam. But it turns out that the muon moves faster with respect to the electrons, although in the opposite direction, than it does with respect to the laboratory.
EXAMPLE 37.12 Kinetic energy and total energy

Calculate the rest energy and the kinetic energy of (a) a 100 g ball moving with a speed of 100 m/s and (b) an electron with a speed of 0.999c.
EXAMPLE 37.12 Kinetic energy and total energy

**MODEL** The ball, with $u \ll c$, is a classical particle. We don’t need to use the relativistic expression for its kinetic energy. The electron is highly relativistic.
EXAMPLE 37.12 Kinetic energy and total energy

SOLVE  a. For the ball, with \( m = 0.10 \text{ kg} \),

\[
E_0 = mc^2 = 9.0 \times 10^{15} \text{ J}
\]

\[
K = \frac{1}{2}mu^2 = 500 \text{ J}
\]

b. For the electron, we start by calculating

\[
\gamma_p = \frac{1}{(1 - u^2/c^2)^{1/2}} = 22.4
\]

Then, using \( m_e = 9.11 \times 10^{-31} \text{ kg} \), we find

\[
E_0 = mc^2 = 8.2 \times 10^{-14} \text{ J}
\]

\[
K = (\gamma_p - 1)E_0 = 170 \times 10^{-14} \text{ J}
\]
EXAMPLE 37.12 Kinetic energy and total energy

ASSESS The ball’s kinetic energy is a typical kinetic energy. Its rest energy, by contrast, is a staggeringly large number. For a relativistic electron, on the other hand, the kinetic energy is more important than the rest energy.