Chapter 40. Wave Functions and Uncertainty

The wave function characterizes particles in terms of the probability of finding them at various points in space. This scanning tunneling microscope image of graphite shows the most probable place to find electrons.

Chapter Goal: To introduce the wave-function description of matter and learn how it is interpreted.
Wave Particle Duality

Be it photons or electrons or neutrons, force fields and matter appear as particulate waves called quantum fields.

How do we reconcile the idea of a particle with wave characteristics?

Our macroscopic notions of objects, of the objective reality of physical properties such as position do not apply to the microscopic world. Quantum objects bear seemingly contradictory wave and particle aspects simultaneously.
Recall the essence of diffraction of any wave. Each aperture acts as a source. Beyond the apertures, the wave amplitudes superpose and the total amplitude exhibits constructive and destructive interference. On the detection plane

\[ A(x) = 2a \cos \left( \frac{\pi d x}{\lambda L} \right) \]

The intensity is proportional to \( A^2 \):

\[ I(x) = C \cos^2 \left( \frac{\pi d x}{\lambda L} \right) \]
The curious thing about quantum wave fields is that the energy and momentum appears in indistinguishable quanta, the number in proportion to the intensity.

Moreover the interference is observed even if the intensity is so low that only one quantum could be in the system at a time. This “particle” evidently feels the whole apparatus, passes through both slits, and interferes with itself in a sense.

That is not like any macroscopic object.
Single Particle Interference

1/30 sec exposure

1 sec exposure

100 sec exposure
The intensity of the light wave is correlated with the probability of detecting particles. That is, particles are more likely to be detected at those points where the wave intensity is high and less likely to be detected at those points where the wave intensity is low.

The probability of detecting a particle in a range $dx$ about a particular point $x$ is directly proportional to the square of the total wave amplitude function at that point:

$$\text{Prob(in } dx \text{ at } x) \propto |A(x)|^2 dx$$
Probability Density

We can define the probability density $P(x)$ such that

$$\text{Prob(in } \delta x \text{ at } x) = P(x) \delta x$$

In one dimension, probability density has SI units of m$^{-1}$. Thus the probability density multiplied by a length yields a dimensionless probability.

**NOTE:** $P(x)$ itself is *not* a probability. You must multiply the probability density by a length to find an actual probability.

The photon probability density is directly proportional to the square of the light-wave amplitude:

$$P(x) \propto |A(x)|^2$$
**FIGURE 40.4** The probability density is analogous to the linear mass density.

The mass of this small segment of string is

$$\text{mass(in } \delta x \text{ at } x) = \mu(x) \delta x$$

The probability that a photon lands in this small segment of the screen is

$$\text{Prob}(\text{in } \delta x \text{ at } x) = P(x) \delta x$$
EXAMPLE 40.1 Calculating the probability density

In an experiment, 6000 out of 600,000 photons are detected in a 1.0-mm-wide strip located at position \( x = 50 \text{ cm} \). What is the probability density at \( x = 50 \text{ cm} \)?

**SOLVE** The probability that a photon arrives at this particular strip is

\[
\text{Prob(in 1.0 mm at } x = 50 \text{ cm}) = \frac{6000}{600,000} = 0.010
\]

Thus the probability density \( P(x) = \frac{\text{Prob(in } \delta x \text{ at } x)}{\delta x} \) at this position is

\[
P(50 \text{ cm}) = \frac{\text{Prob(in 1.0 mm at } x = 50 \text{ cm})}{0.0010 \text{ m}} = \frac{0.010}{0.0010 \text{ m}} = 10 \text{ m}^{-1}
\]
Wavefunctions

For photons, the pair of classical vector fields $E(x)$ and $B(x)$ are associated with the EM field. These are correlated in intensity and orientation by Maxwell’s equations. Often we ignore the polarization and consider just the scalar $E$ field amplitude and call it $A(x)$ or something else.

The generic name “psi (x)” called the wavefunction is used to describe the amplitude at position $x$ of the quantum wave associated with a particle. It is simplest to think of matter waves as on par with the electromagnetic field.
Matter wave functions

EM fields have an electric and a magnetic part

\[ |\psi(x)|^2 \propto E^2(x) + B^2(x) \]

A nonrelativistic matter wave also has two components usually written as a complex valued field and

\[ |\psi(x)|^2 = |\psi_1(x) + i\psi_2(x)|^2 = \psi_1^2(x) + \psi_2^2(x) \]
Normalization

- A photon or electron has to land *somewhere* on the detector after passing through an experimental apparatus.

- Consequently, the probability that it will be detected at *some* position is 100%.

- The statement that the photon or electron has to land *somewhere* on the $x$-axis is expressed mathematically as

\[
\int_{-\infty}^{\infty} P(x) \, dx = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1
\]

- Any wave function must satisfy this **normalization** condition.
**FIGURE 40.8** The area under the probability density curve is a probability.

(a) \[ P(x) = |\psi(x)|^2 \]

The area under the curve between \( x_L \) and \( x_R \) is the probability of finding the particle between \( x_L \) and \( x_R \).

(b) \[ P(x) = |\psi(x)|^2 \]

The total area under the curve must be 1.
De Broglie associates a monochromatic (single wavelength) wave with a particle of momentum $p$

$$p = \frac{\hbar}{\lambda}$$

$$\psi(x) \propto \cos\left(\frac{2\pi x}{\lambda} - \omega t\right) = \cos\left(\frac{px}{\hbar} - \omega t\right)$$

The probability density proportional to $\psi^2$ is at any time is spread in humps uniformly along $x$. We have no idea where the particle is!!
Wave Packets

But we can construct a wave that has a localized probability density so appears more like a classical object – a pulse. The sound of a clap is an example.

A wave that is localized in space at some time is called a wave packet.

It has no unique wavelength so this thing has no unique momentum.
This figure shows how a superposition of waves of wavelength near a central value can sum to a wave packet. An ever wider range is required to make a narrower and sharper packet.
Constructing a Sonic Wave Packets

440 Hz + 439 Hz

440 Hz + 439 Hz + 438 Hz

440 Hz + 439 Hz + 438 Hz + 437 Hz + 436 Hz
Wave Packets

Suppose a single nonrepeating wave packet of duration $\Delta t$ is created by the superposition of many waves that span a range of frequencies $\Delta f$.
Fourier analysis shows that for any wave packet

$$\Delta f \Delta t \approx 1$$

We have not given a precise definition of $\Delta t$ and $\Delta f$ for a general wave packet.
The quantity $\Delta t$ is “about how long the wave packet lasts,” while $\Delta f$ is “about the range of frequencies needing to be superimposed to produce this wave packet.”
EXAMPLE 40.4 Creating RF pulse

EXAMPLE 40.4 Creating radio-frequency pulses
A short-wave radio station broadcasts at a frequency of 10.000 MHz. What is the range of frequencies of the waves that must be superimposed to broadcast a radio-wave pulse lasting 0.800 $\mu$s?

**FIGURE 40.15** A pulse of radio waves.

$E$

$T = 0.100 \mu s$

$\Delta t = 0.800 \mu s$

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EXAMPLE 40.4 Creating radio-frequency pulses

SOLVE The period of a 10.000 MHz oscillation is 0.100 \( \mu s \). A pulse 0.800 \( \mu s \) in duration is 8 oscillations of the wave. Although the station broadcasts at a nominal frequency of 10.000 MHz, this pulse is not a pure 10.000 MHz oscillation. Instead, the pulse has been created by the superposition of many waves whose frequencies span

\[
\Delta f \approx \frac{1}{\Delta t} = \frac{1}{0.800 \times 10^{-6} \text{ s}} = 1.250 \times 10^6 \text{ Hz} = 1.250 \text{ MHz}
\]

This range of frequencies will be centered at the 10.000 MHz broadcast frequency, so the waves that must be superimposed to create this pulse span the frequency range

\[9.375 \text{ MHz} \leq f \leq 10.625 \text{ MHz}\]
The Heisenberg Uncertainty Principle

• The quantity $\Delta x$ is the length or spatial extent of a wave packet.

• $\Delta p_x$ is a small range of momenta corresponding to the small range of frequencies within the wave packet.

• Any matter wave must obey the condition

$$\Delta x \Delta p_x \geq \frac{\hbar}{2} \quad \text{(Heisenberg uncertainty principle)}$$

This statement about the relationship between the position and momentum of a particle was proposed by Heisenberg in 1926. Physicists often just call it the **uncertainty principle**.
The Heisenberg Uncertainty Principle

• If we want to know where a particle is located, we measure its position $x$ with uncertainty $\Delta x$.

• If we want to know how fast the particle is going, we need to measure its velocity $v_x$ or, equivalently, its momentum $p_x$. This measurement also has some uncertainty $\Delta p_x$.

• You cannot measure both $x$ and $p_x$ simultaneously with arbitrarily good precision.

• Any measurements you make are limited by the condition that $\Delta x \Delta p_x \geq \hbar/2$.

• Our knowledge about a particle is inherently uncertain.
Consider the double slit experiment. The uncertainty on the position at the slit location is the slit separation. Try to determine which slit a particular quantum passed through. If you can close one slit but you destroy the interference pattern and have changed the experiment.

With one slit, you know can infer the position uncertainty was the slit width. Diffraction generates a transverse momentum uncertainty

\[ \Delta p_x = p \Delta \theta_x = p \frac{\lambda}{\Delta x} = \frac{\hbar}{\Delta x} \]

Shooting light at the particle doesn’t work either due to the wave nature of light.
Consider a matter wave in a reflective box. The ground state is a standing wave with wavelength twice the box length. The position uncertainty is roughly $L$. The standing wave is a superposition of waves of momentum $+p$ and $-p$ with $p = \frac{h}{\lambda} = \frac{h}{2L}$ so the momentum uncertainty is $2p = \frac{h}{L}$.

$$\Delta p_x \Delta x = 2p \times L = 2 \frac{h}{2L} L = h$$
Quantum particles are creepy. They can have position. They can have momentum. But not both (with arbitrary precision) at the same time.

Other property pairs are complementary in this way including the vector components of their intrinsic angular momentum.

This is unlike what our common sense (approximate) theory of objects tells us.
The uncertainty of a dust particle

**EXAMPLE 40.5  The uncertainty of a dust particle**

A 1.0-\(\mu\)m-diameter dust particle \((m \approx 10^{-15} \text{ kg})\) is confined within a 10-\(\mu\)m-long box. Can we know with certainty if the particle is at rest? If not, within what range is its velocity likely to be found?

\[
\Delta v_x = \frac{\Delta p_x}{m} \approx \frac{h}{2mL} \approx 3.0 \times 10^{-14} \text{ m/s}
\]

This range of possible velocities will be centered on \(v_x = 0 \text{ m/s}\) if we have done our best to have the particle be at rest. Thus all we can know with certainty is that the particle’s velocity is somewhere within the interval \(-1.5 \times 10^{-14} \text{ m/s} \leq v \leq 1.5 \times 10^{-14} \text{ m/s}\).
EXAMPLE 40.5 The uncertainty of a dust particle

**ASSESS** For practical purposes you might consider this to be a satisfactory definition of “at rest.” After all, a particle moving with a speed of $1.5 \times 10^{-14}$ m/s would need $6 \times 10^{10}$ s to move a mere 1 mm. That is about 2000 years! Nonetheless, we can’t know if the particle is “really” at rest.
EXAMPLE 40.6 The uncertainty of an electron

What range of velocities might an electron have if confined to a 0.10-nm-wide region, about the size of an atom?

**SOLVE** The analysis is the same as in Example 40.5. If we know that the electron’s position is located within an interval \( \Delta x \approx 0.1 \text{ nm} \), then the best we can know is that its velocity is within the range

\[
\Delta v_x = \frac{\Delta p_x}{m} \approx \frac{h}{2mL} \approx 4 \times 10^6 \text{ m/s}
\]

Because the *average* velocity is zero, the best we can say is that the electron’s velocity is somewhere in the interval \(-2 \times 10^6 \text{ m/s} \leq v \leq 2 \times 10^6 \text{ m/s}\). It is simply not possible to know the electron’s velocity any more precisely than this.

The velocity of electrons in atoms is about 1% of light speed.