Chapter 28. Gauss’s Law

Using Gauss’s law, we can deduce electric fields, particularly those with a high degree of symmetry, simply from the shape of the charge distribution. The nearly spherical shape of the girl’s head determines the electric field that causes her hair to stream outward.

Chapter Goal: To understand and apply Gauss’s law.
The electric field inside a conductor in electrostatic equilibrium is

A. uniform.
B. zero.
C. radial.
D. symmetric.
The electric field inside a conductor in electrostatic equilibrium is

A. uniform.

B. zero.  

C. radial.

D. symmetric.

☑ B. zero.
The Electric Flux

The electric flux measures the flow of electric field. The flux through a small surface of area $A$ whose normal to the surface is tilted at angle $\theta$ from the field is:

$$\Phi_e = E_{\perp}A = EA \cos \theta$$

Or using the dot-product:

$$\Phi_e = \vec{E} \cdot \vec{A} \quad \text{(electric flux of a constant electric field)}$$

We have assumed $A$ is so small that $E$ is uniform within it.
When the electric field is parallel to the surface so perpendicular to the area vector, the flux thru the surface vanishes. (angle is 90 degrees). When the electric field is perpendicular to the surface so parallel to the area vector, the flux thru the surface is maximal and $E \cdot A$. (angle is 0 degrees).
EXAMPLE 28.1 The electric flux inside a parallel-plate capacitor

QUESTION:

Two 100 cm$^2$ parallel electrodes are spaced 2.0 cm apart. One is charged to +5.0 nC, the other to −5.0 nC. A 1.0 cm $\times$ 1.0 cm surface between the electrodes is tilted to where its normal makes a 45° angle with the electric field. What is the electric flux through this surface?
EXAMPLE 28.1 The electric flux inside a parallel-plate capacitor

SOLVE  In Chapter 27, we found the electric field inside a parallel-plate capacitor to be

\[
E = \frac{Q}{\varepsilon_0 A_{\text{plates}}} = \frac{5.0 \times 10^{-9} \text{ C}}{(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)(1.0 \times 10^{-2} \text{ m}^2)}
\]

\[
= 5.65 \times 10^4 \text{ N/C}
\]

A 1.0 cm \( \times \) 1.0 cm surface has \( A = 1.0 \times 10^{-4} \text{ m}^2 \). The electric flux through this surface is

\[
\Phi_e = \vec{E} \cdot \vec{A} = EA \cos \theta
\]

\[
= (5.65 \times 10^4 \text{ N/C})(1.0 \times 10^{-4} \text{ m}^2) \cos 45^\circ
\]

\[
= 4.0 \text{ N m}^2/\text{C}
\]
The Electric Flux for a finite surface

When the electric field is non uniform or the surface curved, approximate the total flux by adding the contributions of little pieces. This defines the surface integral.

\[
\delta \Phi_E^i = E_i \delta A_i \cos \theta = \vec{E}_i \cdot \delta \vec{A}_i
\]

\[
\Phi_e = \int_{\text{surface}} \vec{E} \cdot d\vec{A}
\]
The Electric Flux through a Closed Surface

The electric flux through a closed surface is

\[ \Phi_e = \oint \vec{E} \cdot d\vec{A} \]

The electric flux is still the summation of the fluxes through a vast number of tiny pieces, pieces that now cover a closed surface.

**NOTE:** A closed surface has a distinct inside and outside. The area vector \( dA \) is defined to always point *toward the outside*. 
Gauss’s Law

For any closed surface enclosing total charge \( Q_{in} \), the net electric flux through the surface is

\[
\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}
\]

This result for the electric flux is known as Gauss’s Law.

Gauss’s law follows from Coulomb’s law but is more general. It applies when charges moves and Coulomb’s law is invalid.
Using Gauss’s Law

1. Gauss’s law applies only to a *closed* surface, called a Gaussian surface.

2. A Gaussian surface is not a physical surface. It is an imaginary, mathematical surface in the space surrounding one or more charges.

3. We can’t generally find the electric field from Gauss’s law alone. We can apply Gauss’s law when from symmetry and superposition, we already can guess the *shape* of the field.
Gauss’s Law for a point charge

- Consider a closed spherical surface, centered on a positive point charge.
- The E-field is perpendicular to surface, directed outward.
- E-field magnitude on sphere:
  \[ E = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \]

- All surface elements have the same electric field:
  \[ \Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E dA = E \oint dA = EA = \left( \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \right) \left( 4\pi R^2 \right) = \frac{q}{\epsilon_0} \]
Gauss’s Law for a point charge
Two spheres of radius $R$ and $2R$ are centered on the positive charges of the same value $q$. The electric flux through the spheres compare as:

A) $\text{Flux}(R) = \text{Flux}(2R)$
B) $\text{Flux}(R) = 2\times \text{Flux}(2R)$
C) $\text{Flux}(R) = (1/2)\times \text{Flux}(2R)$
D) $\text{Flux}(R) = 4\times \text{Flux}(2R)$
E) $\text{Flux}(R) = (1/4)\times \text{Flux}(2R)$
Gauss’s Law for a point charge

Two spheres of the same size enclose different positive charges, q and 2q. The flux through these spheres compare as

A) \(\text{Flux}(A) = \text{Flux}(B)\)

B) \(\text{Flux}(A) = 2\text{Flux}(B)\)

C) \(\text{Flux}(A) = (1/2)\text{Flux}(B)\)

D) \(\text{Flux}(A) = 4\text{Flux}(B)\)

E) \(\text{Flux}(A) = (1/4)\text{Flux}(B)\)
General Gauss’s Law

\[ \Phi_E = \oint E \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \]

- Works for any closed surface, not only sphere
- Works for any charge distribution (by superposition)

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Field of a sphere of charge

**FIGURE 28.23** A spherical Gaussian surface surrounding a sphere of charge.

- **Gaussian surface**
- **Sphere of total charge** $Q$
- **$E$** is everywhere perpendicular to the surface.
**EXAMPLE 28.3 Outside a sphere of charge**

**SOLVE** Gauss’s law is

\[ \Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \]

To calculate the flux, notice that the electric field is everywhere perpendicular to the spherical surface. And although we don’t know the electric field magnitude \( E \), spherical symmetry dictates that \( E \) must have the same value at all points equally distant from the center of the sphere.
EXAMPLE 28.3 Outside a sphere of charge

Thus we have the simple result that the net flux through the Gaussian surface is

\[ \Phi_e = EA_{\text{sphere}} = 4\pi r^2 E \]

where we used the fact that the surface area of a sphere is \( A_{\text{sphere}} = 4\pi r^2 \). With this result for the flux, Gauss’s law is

\[ 4\pi r^2 E = \frac{Q}{\epsilon_0} \]

Thus the electric field at distance \( r \) outside a sphere of charge is

\[ E_{\text{outside}} = \frac{1}{4\pi \epsilon_0} \frac{Q}{r^2} \]

Or in vector form, making use of the fact that \( \vec{E} \) is radially outward,

\[ \vec{E}_{\text{outside}} = \frac{1}{4\pi \epsilon_0} \frac{Q}{r^2} \hat{r} \]

where \( \hat{r} \) is a radial unit vector.
Line charge

- Field direction: radially out from line charge
- Gaussian surface:
  - Cylinder of radius $r$
- Area where $\vec{E} \cdot d\vec{A} \neq 0$:
  - $2\pi rL$
- Value of $\vec{E} \cdot d\vec{A}$ on this area:
  - $E$
- Flux thru Gaussian surface:
  - $E 2\pi rL$
- Charge enclosed:
  - $\lambda L$

Gauss’ law: \[
E 2\pi rL = \frac{\lambda L}{\varepsilon_o} \Rightarrow E = \frac{1}{2\pi\varepsilon_o} \frac{\lambda}{r}
\]
Plane of charge

- Field direction: perpendicular to plane
- Gaussian surface:
  - Cylinder of radius $r$
- Area where $\vec{E} \cdot d\vec{A} \neq 0$:
  - $2\pi r^2$
- Value of $\vec{E} \cdot d\vec{A}$ on this area:
  - $E$
- Flux thru Gaussian surface:
  - $E2\pi r^2$
- Charge enclosed:
  - $\eta \pi r^2$

- Gauss' law: $E2\pi r^2 = \eta \pi r^2 / \varepsilon_o \Rightarrow E = \frac{\eta}{2\varepsilon_o}$
Conductors in Electrostatic Equilibrium

The electric field is zero at all points within a conductor in electrostatic equilibrium. Else charge carriers would move.
Apply Gauss’s Law to a pill box spanning the surface of a conductor. The charge enclosed is \( \eta A \) where \( \eta \) is the surface charge density of the conductor. The total outward flux is \( EA \) from the outer disk parallel to the local surface and must be the charge enclosed.

\[
\vec{E}_{\text{surface}} = \left( \frac{\eta}{\varepsilon_0}, \text{perpendicular to surface} \right)
\]

**FIGURE 28.30** A Gaussian surface extending through the surface of the conductor has a flux only through the outer face.
EXAMPLE 28.7 The electric field at the surface of a charged metal sphere

QUESTION:

EXAMPLE 28.7 The electric field at the surface of a charged metal sphere
A 2.0-cm-diameter brass sphere has been given a charge of 2.0 nC. What is the electric field strength at the surface?
EXAMPLE 28.7 The electric field at the surface of a charged metal sphere

**SOLVE** We can solve this problem in two ways. One uses the fact that a sphere is the one shape for which any excess charge will spread out to a *uniform* surface charge density. Thus

\[
\eta = \frac{q}{A_{\text{sphere}}} = \frac{q}{4\pi R^2} = \frac{2.0 \times 10^{-9} \text{ C}}{4\pi (0.010 \text{ m})^2} = 1.59 \times 10^{-6} \text{ C/m}^2
\]

From Equation 28.20, we know the electric field at the surface has strength

\[
E_{\text{surface}} = \frac{\eta}{\epsilon_0} = \frac{1.59 \times 10^{-6} \text{ C/m}^2}{8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2} = 1.8 \times 10^5 \text{ N/C}
\]
EXAMPLE 28.7 The electric field at the surface of a charged metal sphere

Alternatively, we could have used the result, obtained earlier in the chapter, that the electric field strength outside a sphere of charge $Q$ is $E_{\text{outside}} = Q_{\text{in}}/(4\pi\varepsilon_0 r^2)$. But $Q_{\text{in}} = q$ and, at the surface, $r = R$. Thus

$$E_{\text{surface}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} = (9.0 \times 10^9 \text{ Nm}^2/\text{C}^2) \frac{2.0 \times 10^{-9} \text{ C}}{(0.010 \text{ m})^2}$$

$$= 1.8 \times 10^5 \text{ N/C}$$

As we can see, both methods lead to the same result.
A uniformly charged rod has a finite length \( L \). The rod is symmetric under rotations about the axis and under reflection in any plane containing the axis. It is \textit{not} symmetric under translations or under reflections in a plane perpendicular to the axis other than the plane that bisects the rod. Which field shape or shapes match the symmetry of the rod?

A. c and e  
B. a and d  
C. e only  
D. b only  
E. none of the above
A uniformly charged rod has a finite length $L$. The rod is symmetric under rotations about the axis and under reflection in any plane containing the axis. It is not symmetric under translations or under reflections in a plane perpendicular to the axis other than the plane that bisects the rod. Which field shape or shapes match the symmetry of the rod?

A. c and e
B. a and d
C. e only
D. b only
E. none of the above

The correct answer is B. a and d.

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This box contains

A. a net positive charge.
B. a net negative charge.
C. a negative charge.
D. a positive charge.
E. no net charge.
This box contains

A. a net positive charge.
B. a net negative charge.  
C. a negative charge.
D. a positive charge.
E. no net charge.
The total electric flux through this box is

A. 6 Nm$^2$/C.
B. 4 Nm$^2$/C.
C. 2 Nm$^2$/C.
D. 1 Nm$^2$/C.
E. 0 Nm$^2$/C.
The total electric flux through this box is

A. 6 Nm$^2$/C.
B. 4 Nm$^2$/C.
C. 2 Nm$^2$/C.  [Correct Answer]
D. 1 Nm$^2$/C.
E. 0 Nm$^2$/C.