Problem 1: Quantum Theory of the Hydrogen Atom

We have seen that the most probable radial position of an electron in the ground state of hydrogen is the Bohr radius, $a_0$. How does the expectation (average) value of the radial position of the electron in this state, $< r > = \int r |\psi_{100}|^2 \, dr$, compare to this value? Recall that $\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$.

Solution:

The average value of the radial position of the electron is

$$< r > = \int r \psi_{100}^* \psi_{100} \, d^3r = \int_0^\infty \int_0^{2\pi} \int_0^\pi r |\psi_{100}|^2 r^2 \sin \theta \, d\theta \, d\phi$$

As we have

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}, \quad < r > = \int_0^\infty \frac{4\pi}{\pi a_0^3} r^3 e^{-2r/a_0} \, dr = 4a_0 \int_0^\infty u^3 e^{-2u} \, du = \frac{3}{2} a_0,$$

which is clearly larger than the Bohr radius.
Problem 2. T&L 7.40 (Spin-orbit effect)
The prominent yellow doublet lines in the spectrum of sodium (see figure) result from transitions from the 3P\(_{3/2}\) and 3P\(_{1/2}\) states to the ground state. The wavelengths of these 2 lines are 589.59 nm and 588.99 nm.

(a) Calculate the energies in eV of the photons corresponding to these wavelengths.
(b) The difference in energy of these photons equals the difference in energy \(\Delta E\) of the 3P\(_{3/2}\) and 3P\(_{1/2}\) states. This energy difference is due to the spin-orbit effect. Calculate \(\Delta E\).
(c) Estimate the internal B-field of that the 3p electron feels from the energy difference found in (b).

Solution:

\[ E = \frac{hc}{\lambda} \]

\( E_{3/2} = \frac{1240 \text{ nm} \cdot \text{eV}}{589.99} = 2.10505 \text{ eV} \)
\( E_{1/2} = \frac{1240 \text{ nm} \cdot \text{eV}}{589.59} = 2.10291 \text{ eV} \)

\( \Delta E = E_{3/2} - E_{1/2} = 2.14 \times 10^{-3} \text{ eV} \)

Remember that

\[ \mu_B = \frac{e\hbar}{2m_e} = 9.2740154 \times 10^{-24} J / \text{T} = 5.7883826 \times 10^{-5} \text{ eV} / \text{T} \]

Bohr magneton
(c) $\Delta E = 2\mu_B B \Rightarrow B = \frac{\Delta E}{2\mu_B} = \frac{2.14 \times 10^{-3} eV}{2 \times 5.79 eV / T} = 18.5 T$

**Problem 2. T&L 7.65 (Stern and Gerlach experiment)**

In a Stern-Gerlach experiment H atoms in their ground state move with speed $v_x = 14.5$ km/s. The magnetic field is in the z direction and its maximum gradient is given by $dB_z/dz = 600$ T/m.

(a) Find the maximum acceleration of the H atoms.

(b) If the region of the magnetic field extends over a distance $\Delta x = 75$ cm and there is an additional 1.25 m from the edge of the field to the detector find the maximum distance between the 2 lines on the detector. [Consider that the mass of the H atom is about the mass of the proton $= 1.67 \times 10^{-27}$ kg.]

**(b) is optional**

![Diagram of Stern-Gerlach experiment]

**(a)**

![Diagram of atomic beam and collector plate]

**(b)**

![Diagram of magnetic field lines]

**(c)**

![Diagram of magnetic field lines]

**Solution:**

Since the atoms are in the ground state $\ell = 0$ and so $m=0$ and so only the spin contributes to the splitting of the two lines on the detector. The force and potential energy are related by:

$$\vec{F} = -\nabla U = -\nabla (-\vec{\mu} \cdot \vec{B}) \Rightarrow F_z = m_s g_s \mu_B \frac{dB}{dz} \Rightarrow a_z = m_s g_s \mu_B \frac{dB}{dz} / m_H = \mu_B \frac{dB}{dz} / m_H = \frac{9.27 \times 10^{-24} \times 600}{1.67 \times 10^{-27}} = 3.33 \times 10^6$ \text{m/s}^2$$

where we used $g_s = 2$ and $|m_s| = \frac{1}{2}$.

(b) In the region of the magnetic field there is a force and the motion is uniformly accelerated along the z axis:

$$z_1 = \frac{1}{2} a_z t_1^2$$
Along the x axis there is no force so the atom moves at constant velocity of $v_x = 14.5 \text{ km/s}$ in the region in the magnetic field of length $\Delta x = 75 \text{ cm}$:

$$t_1 = \frac{0.75}{14.5 \times 10^3} = 5.2 \times 10^{-5} \text{s}$$

So the deflection along z when the beam exits the magnetic field region is:

$$z_1 = \frac{1}{2} a_z t_1^2 = 4.5 \times 10^{-3} \text{ m}.$$ 

When the atom travels in the additional 1.25 m its $v_z$ velocity as it leaves the magnet is $v_z = a_z t_1$ which adds an additional z deflection of:

$$z_2 = v_z t_2 = a_z t_1 \frac{1.25m}{v_x} = 3.33 \times 10^6 \text{ m/s}^2 \times 5.2 \times 10^{-5} \text{s} \times \frac{1.25m}{14.5 \times 10^3} = 1.49 \times 10^2 \text{ m}$$

So each line will be deflected by $z_1 + z_2 = 2.0 \text{ cm}$. 