Lecture 26

An imperfect Bose gas

Brief summary of ideal Bose gas

$T=0$:

$N=0$, all particles are in the condensate

\[ p=0 \quad \Rightarrow \quad E = \frac{p^2}{2m} \]

$T<T_0$:

$T_0 = \frac{\hbar^2}{4\pi^2 m} \left( \frac{N}{V} \right)^{2/3}$

Two kinds of particles:

\[
\begin{align*}
N_{>0} &= N \left( \frac{T}{T_0} \right)^{3/2} \\
N_{=0} &= N \left( 1 - \left( \frac{T}{T_0} \right)^{3/2} \right)
\end{align*}
\]

\[ \text{He}^4: \quad \downarrow \]

$T_{\text{BEC}} = 3.14 \text{K}$

λ-transition

Since He$^4$ atoms obey Bose statistics, it is natural to assume that this transition is a Bose-Einstein condensation.

If we compare $T_{\text{BEC}}$ for ideal gas and experimental result for He$^4$, we will see the difference.

$T_{\text{BEC}} = 2.18 \text{K}$ but also for ideal Bose gas
He is a weakly interacting Bose gas.

Let us consider He as weakly interacting Bose gas. This assumption is justified as He is a noble gas.

Let us perform our analysis using second quantization form.

\[ a_p^+ \equiv \text{creation operator with momentum } p \]

\[ a_p \equiv \text{annihilation operator} \]

Let us assume that He can be considered as a gas of \( N \) identical bosons without spin.

\[ [a_p a_p^+] = a_p a_p^+ - a_p^+ a_p = 1 \]

\[ a_p^+ a_p = n_p = \text{number of particles} \]

The Hamiltonian is given by

\[ H = \sum \frac{p^2}{2m} \cdot n_p + U, \text{ where} \]

\( U \) is the energy of boson interactions.

If we consider interactions only due to binary collisions, when

\[ U = \frac{u_0}{2V} \sum a_p^+ a_{p_2} a_p a_{p_4} \delta(p_1 + p_2 - p_3 - p_4), \]

where

\[ u_0 = \int u(r) d^3r. \]

For example, \( U(r) = u_0 e^{-r/\rho_0} \).
Interactions are weak when the density of bosons is small.

\[ d = \sqrt[3]{\frac{\hbar}{m}} \quad a \equiv \text{scattering length} \]

\[ a \ll d \implies a \left( \frac{N}{V} \right)^{\frac{1}{3}} \ll 1 \]

\[ u_0 = \frac{4 \pi \hbar^2}{m} a \]

\[ a \ll \left( \frac{V}{N} \right)^{\frac{1}{3}} \]

If interactions are weak, we can consider them perturbatively. The starting point for the application of the perturbation theory is the following:

At \( t=0 \), all particles of the ideal gas are in the condensate.

\[ \begin{cases} N_{p=0} = N_0 = N \\ N_{p \neq 0} = 0 \end{cases} \]

In nearly ideal gas, \( N_{p=0} \approx N \)

\[ N \sim a_0^+ a_0 \quad \implies \quad a_0 \sim \sqrt{N}. \quad \text{This is because} \]

\[ [a_0 a_0^+] = a_0 a_0^+ - a_0^+ a_0 = 1 \ll a_0 \quad \text{, i.e.} \]

as the commutator \([a_0 a_0^+] \propto O(N)\), then \( a_0, a_0^+ \) can be considered as numbers. This is a big simplification, because we can consider \( a_0, a_0^+ \sim \sqrt{N} \) as a small parameter.

Note that for weakly interacting Bose gas,

\[ a_0^+ a_0 + \sum_p a_p^+ a_p = N \]

\[ \sum_p a_p^+ a_p = N \rho \ll N. \quad \text{However,} \quad \sum_p \langle a_p^+ a_p \rangle \sim N \quad \text{after averaging} \]
Let see how it will work.

Perturbation treatment of the interacting term gives

at the zeroth order (substitute all operators with $a_0$)

$$\sum_{p_1 p_2} a^+_1 a_{p_2} a_{p_1} a_{p_2} a_{p_1} = a_0^4$$

at the first order (substitute only 3 operators with $a_0$)

$$\Rightarrow$$ zero since the first order term cannot satisfy the conservation of momenta.

at the second order (substitute 2 operators with $a_0$)

$$a_0^2 \sum_{p \neq 0} (a^+_p a^-_p + a^+_p a^-_{-p} + 4 a^+_p a^-_p)$$

from combinatorics

the creation of a pair of particles from the vacuum.

The zeroth + the second order give us

$$a_0^4 + 4 a_0^2 \sum_{p \neq 0} (a^+_p a^-_p + a^+_p a^-_{-p}) + a_0^2 \sum_{p \neq 0} (a^+_p a^-_p + a^+_p a^-_{-p})$$

$$\sim N^2$$

Now the Hamiltonian can be written as

$$H = E_0 + \frac{\hbar^2}{2m} \sum_p \frac{p^2}{2m} a^+_p a_p + \mu_0 N \sum_{p \neq 0} (a^+_p a^-_{-p} + a^+_p a^-_p)$$

$$E_0 = \frac{N^2}{2V} \mu_0 - \text{the ground state energy of weakly interacting Bose gas.}$$
Note, \( E_0 = 0 \), contrary to the ideal gas, those energy is zero at \( T=0 \).

The third term gives corrections to the ground state energy but also renormalizes the spectrum of weakly excited states of Bose gas.

In order to find the excitation spectrum, we diagonalize the Hamiltonian using the Bogoliubov transformation. Let's rewrite the Hamiltonian in a standard form.

\[
H = \sum_p \varepsilon_p a_p^+ a_p + \frac{\hbar^2}{2m} \sum_p \left( a_p^+ a_{-p}^+ + a_p a_{-p} \right)
\]

Bogoliubov transformation (BT) is given by

\[
\begin{align*}
    a_p &= \nu_p b_p + \tilde{\nu}_p b_{-p}^+ \\
    a_p^+ &= \nu_p b_p^+ + \tilde{\nu}_p b_{-p}
\end{align*}
\]

\( b_p \) are operators of quasiparticles, which are bosons which create or annihilate quasiparticle on the eigenstates with energy \( \varepsilon_p \).

In physics, the term quasiparticles (QP) is often used to describe weakly interacting original particles. QP usually describe collective modes, they correspond to a collective motion of the system as a whole.

He we need to find coefficients \( \nu_p \) and \( \tilde{\nu}_p \).

In the diagonalized form, the Hamiltonian is

\[
H = \sum_p \varepsilon_p b_p^+ b_p
\]
\[
\begin{align*}
\{ b_p, H \} & = \sum_{p'} E_{p'} \{ b_p, b_{p'}^+ b_{p'} \} = \sum_{p'} E_{p'} \left[ \{ b_p, b_{p'}^+ \} b_{p'} + \right. \\
& \left. \tilde{\delta}_{p, p'} \right] \\
& + \left[ b_p, b_{p'}^+ \right] b_{p'}^+ = E_p b_p \\
& = 0 \\
\{ b_{p'}^+, H \} & = \sum_{p'} E_{p'} \{ b_{p'}^+, b_{p'}^-, b_{p'} \} = \sum_{p'} E_{p'} \left[ \{ b_{p'}^+ b_{p'} \} b_{p'} + \right. \\
& \left. \tilde{\delta}_{p, p'} \right] \\
& + \left[ b_p b_{p'}^\dagger \right] b_{p'}^+ = -E_{p'} b_{p'} \\
& = -\delta_{p, p'} \\
\end{align*}
\]

From the other hand, using Bogoliubov transformation in the inverse form
\[
\begin{align*}
b_p &= u_p a_p - v_p a_p^+ \\
b_{p'}^+ &= u_{p'} a_{p'}^+ - v_{p'} a_{p'} \\
\end{align*}
\]
we get
\[
\begin{align*}
\{ b_p, H \} &= \left[ (u_p a_p - v_p a_p^+) \right. \\
& \left. \sum_{p'} \left( a_{p'}^+ a_{p'} + \frac{b_{p'}^+}{\tilde{\delta}} (a_{p'} a_{p'} + a_{p'} a_{p'}) \right) \right] \\
& = E_p b_p = E_p (u_p a_p - v_p a_p^+) \quad (1) \\
\end{align*}
\]
and
\[
\begin{align*}
\{ b_{p'}^+, H \} &= \left[ (u_{p'} a_{p'}^+ - v_{p'} a_{p'}) \right. \\
& \left. \sum_{p'} \left( a_{p'}^+ a_{p'} + \frac{b_{p'}}{\tilde{\delta}} (a_{p'} a_{p'} + a_{p'} a_{p'}) \right) \right] \\
& = E_{p'} b_{p'} = -E_p (u_{p'} a_{p'}^+ - v_{p'} a_{p'}) \quad (1') \\
\end{align*}
\]
To compute the commutators in the R.H.S. of these equations, we use

\[
[a_p, a_p^+, a_{-p}^+] = \delta_{pp'} a_{-p}^+
\]

\[
[a_{-p}, a_p^+, a_{-p}^+] = \delta_{pp'} a_p^+
\]

\[
[a_p^+, a_p^+, a_{-p}] = -\delta_{pp'} a_{-p}
\]

\[
[a_{-p}^+, a_p^+, a_{-p}] = -\delta_{pp'} a_p
\]

\[
[a_p^+, a_p, a_{-p}] = -\delta_{pp'} a_p^+
\]

\[
[a_{-p}^+, a_p^+, a_{-p}] = -\delta_{pp'} a_p^+
\]

\[
[a_p, a_p^+, a_{-p}] = \delta_{pp'} a_p
\]

\[
[a_{-p}, a_p^+, a_{-p}] = \delta_{pp'} a_{-p}
\]

We get

\[
(!!) : \quad u_p a_p a_{-p} + \frac{u_p B_p}{2} a_{-p}^+ + u_p A_p a_{-p}^+ + \frac{u_p B_p a_p}{2} = \frac{E_p u_p a_p}{2} - \frac{E_p u_p a_{-p}^+}{2}
\]

\[
(!!) : \quad -u_p A_p a_p^+ - \frac{u_p B_p a_{-p}}{2} - u_p A_p a_{-p} - \frac{u_p B_p a_{-p}^+}{2} = -\frac{E_p u_p a_p^+}{2} + \frac{E_p u_p a_{-p}}{2}
\]

\[
\begin{cases}
(u_p A_p + \frac{u_p B_p}{2} - E_p u_p) a_{-p}^+ + (\frac{u_p B_p + u_p A_p}{2} - E_p u_p) a_{-p} = 0 \\
(\frac{u_p B_p + u_p A_p + E_p u_p}{2}) a_{-p} + (u_p A_p + u_p \frac{B_p}{2} - E_p u_p) a_{-p} = 0
\end{cases}
\]
\[ U_p = \sqrt{\frac{A_p + E_p}{2E_p}} \]
\[ V_p = -\frac{B_p}{|B_p|} \sqrt{\frac{A_p - E_p}{2E_p}} \]
\[ E_p = \sqrt{A_p^2 - B_p^2} \]

The Bogoliubov transformation is very useful in general and can be applied for diagonalization of various quadratic forms.

Bose gas + interactions

\[ A_p = \frac{p^2}{2m} + 2\lambda = \frac{p^2}{2m} + \frac{N}{V} u_0 \quad \mid \quad \beta = \frac{N}{2V} u_0 \]
\[ B_p = 2\lambda = \frac{N}{V} u_0 \]

\[ E_p = \sqrt{A_p^2 - B_p^2} = \sqrt{\left(\frac{p^2}{2m} + 2\lambda\right)^2 - (2\lambda)^2} = \sqrt{\frac{p^2}{2m}(4\lambda + \frac{p^2}{2m})} \]

When \( p \to 0 \) \( \left(\frac{p^2}{2m} \ll 4\lambda\right) \), \[ E_p = U P \]

\[ U = \sqrt{\frac{2\lambda}{m}} \]

\( u \) is the sound velocity. We got a very important result. At small \( p \) the excitations
of the weakly interacting Bose gas have linear dispersions. Sound waves are the low-energy excitations.

$b_p$ and $b_p^+$ are operators of $\hat{\pi}$. At the ground state ($T=0$),

$$\langle b_p^+ b_p \rangle = 0 \quad \text{no quasiparticles}$$

$$\hat{n}_p = \frac{1}{\epsilon_p/T} = \langle b_p^+ b_p \rangle \quad \text{at finite } T.$$

$$\mu = 0$$

For the original bosons, we have

$$\hat{n}_p = \langle \hat{a}_p^+ \hat{a}_p \rangle = \frac{\epsilon_p - E_p}{2E_p} + \frac{\epsilon_p}{2E_p} \hat{n}_p$$

At $T=0$, 

$$\langle \hat{a}_p^+ \hat{a}_p \rangle = \frac{\epsilon_p - E_p}{2E_p} \neq 0$$

This is because due to interactions at $T=0$ not all particles are in the condensate.