Summary: we consider imperfect Bose gas.

Bogoliubov transformation:

BT named after Nikolay Bogoliubov, is a unitary transformation using either canonical commutation relations (for bosons) or anticommutation relations (for fermions). The BT is used to diagonalize Hamiltonians in order to obtain eigenmodes and eigenenergies.

\[
H = \sum_p \left( a_p a_p^+ b_p^+ + \frac{B_p}{2} (a_p a_{-p} + a_{-p}^+ a_p^+) \right) \Rightarrow \sum_p E_p b_p^+ b_p
\]

**Bosons**

\[
\begin{align*}
\{a_p = & u_p b_p + \nu_p b_{-p}^+ \\
\{a_p^+ = & u_p b_p^+ + \nu_p b_{-p}
\end{align*}
\]

\[
u_p^2 - \nu_{-p}^2 = 1
\]

\[
\begin{align*}
[b_p, H] &= E_p b_p \\
[(u_p a_p - \nu_p a_{-p}^+), \sum_{p'} \left( a_{p'} a_{p'}^+ a_p + \frac{B_{p'}}{2} (a_{p'} a_{-p'} + a_{-p'}^+ a_{p'}) \right)] &= 0 \\
[b_p^+, H_p] &= -E_p b_p^+ \\
[(u_p a_p^+ - \nu_p a_{-p}), \sum_{p'} \left( a_{p'} a_{p'}^+ a_p + \frac{B_{p'}}{2} (a_{p'} a_{-p'} + a_{-p'}^+ a_{p'}) \right)] &= 0
\end{align*}
\]
\[ u_p = \sqrt{\frac{A_p + E_p}{2E_p}} \]

\[ v_p = -\frac{B_p}{18po} \sqrt{\frac{A_p - E_p}{2E_p}} \]

\[ E_p = \sqrt{A_p^2 - B_p^2} \]

**Bose gas + interactions**

\[ A_p = \frac{p^2}{2m} + \frac{N}{V} u_0 = \frac{p^2}{2m} + 2\lambda \]

\[ B_p = \frac{N}{V} u_0 = 2\lambda \]

\[ \Rightarrow E_p = \sqrt{A_p^2 - B_p^2} = \left(\frac{p^2}{2m} + 2\lambda\right)^2 - \left(2\lambda\right)^2 = \]

\[ = \frac{p^2}{2m} \left(4\lambda + \frac{p^2}{2m}\right) \]

when \( p \to 0 \left(\frac{p^2}{2m} \ll 4\lambda\right) \)

\[ E_p \approx p \sqrt{\frac{2\lambda}{m}} = P U - \text{linear dependence on } p \]

\[ U = \sqrt{\frac{2\lambda}{m}} \] is the sound velocity; at small \( p \), the excitations of weakly interacting Bose gas are sound waves (with linear dispersion)
Ground state properties of weakly interacting Bose gas.

\[ T = 0 \]

\[ \hat{\mathbf{n}}_p = \langle b_p^+ b_p \rangle = 0 \quad \text{no quasiparticles} \]

( no excitations )

\[ \eta_p = \langle \alpha_p^+ \alpha_p \rangle = \left( \langle b_p^+ b_p + \nu_p b_p^+ \nu_p^\dagger b_p^\dagger \rangle \right) = \]

\[ = \nu_p^2 \langle b_p^+ b_p \rangle + \nu_p^2 \langle b_p^+ b_p \rangle + \]

\[ = \nu_p^2 \langle b_p^+ b_p \rangle + \nu_p^2 \langle b_p^+ b_p \rangle \]

\[ = \nu_p^2 \langle 1 - b_p^+ b_p \rangle = \nu_p^2 - \nu_p^2 \langle b_p^+ b_p \rangle \]

\[ \Rightarrow \eta_p = \langle \alpha_p^+ \alpha_p \rangle = \nu_p^2 = \frac{4}{2 E_p} \neq 0 \]

The number of original bosons is non-zero. Not all particles are in the condensate.

\[ T \neq 0 \]

\[ \hat{\mathbf{n}}_p = \frac{1}{E_p/\hbar - 1} = \langle b_p^+ b_p \rangle \]

\[ \eta_p = \frac{\hbar^2}{2 E_p} + \frac{\hbar^2}{E_p} \hat{\mathbf{n}}_p \]

Liquid \(^4\)He

Liquid \(^4\)He

\[ \begin{align*}
\varepsilon & = \frac{\hbar^2}{2m} (\frac{p^2}{2m} - 4) \\
\varepsilon_{\text{cut}} & = \frac{\hbar^2}{2m} \rho_0^2 \\
\rho & < \rho_0 \]

Sound waves (phonons)
At large momenta, the dispersion relation is more complicated.

Rotons are different quasiparticles.

Around $p_0$: $E = \Delta + \frac{(p - p_0)^2}{2m^*}$ - roton spectrum.

$\Delta \equiv E(p_0)$.

For $^4$He $\Delta = 8.7$ K

$m^* = 0.16 m (^4\text{He})$

Rotons have Boltzmann distribution.

As $\frac{\Delta}{T} \gg 1$ at low-$T$,

$$n_{\text{rot}} \sim e^{-\frac{\Delta}{kT}}$$

One can compute the contribution of both phonons and rotons into specific heat and free energy.

$$C_{\text{phon}} \sim T^3$$

$$C_{\text{rot}} = \frac{3}{2} h + \frac{\Delta}{T} + \left(\frac{\Delta}{T}\right)^2$$

$$N_{\text{rot}} = \frac{2m^* T}{2m^* T} \frac{p_0^2 V}{\left(2m^* T \right)^{3/2} T^3} e^{\frac{-\Delta}{kT}}$$

$$F_{\text{rot}} = -T N_{\text{rot}}$$
At $T < T_{clos} \approx 0.8 K$, the specific heat is mainly determined by phonons.

At $T > T_{clos}$, the specific heat is mainly determined by rotons.

Brief conclusion of what one has to know about Bose liquid.

1) $T=0$, condensate ($T < T_{clos}$)

2) excitations are bosons with linear dispersion $\varepsilon = \hbar \omega_p$, and $\mu = 0$.

$$\hat{N}_p = \langle b_p^+ b_p \rangle = \frac{1}{e^{\varepsilon/\hbar} - 1}$$

3) at low-$T$, "phonons" are nearly noninteracting quasiparticles

4) in He$^4$, there is another kind of quasiparticles - rotons.