Vortices in superfluid

If one starts to rotate a vessel containing a superfluid, initially the liquid remains at rest. However, starting from a certain critical angular velocity, a vortex will be formed in a liquid: a circular motion of the superfluid around a certain line which is called vortex core.

In a cylindrical vessel vortices start at the bottom and go all the way to the top. They cannot be interrupted and cannot simply end inside the liquid.

Let us show that the circulation of velocity around a vortex should be quantized. The story of quantized vortices provides an important insight into the problem of rotohrons in superfluids. Quantized vortices were first predicted by Onsager and Feynman. In fact, the superfluid cannot rotate. In usual rigid bodies,
the tangential velocity is \[ \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \]

Next important thing is that

\[ Q_s (r, t) = |\psi_0 (r, t)|^2 \quad \Rightarrow \quad N_0 = \int |\psi_0 (r, t)|^2 \, dr \, dt \]

\[ Q_s = \frac{N_0}{\sqrt{\gamma}} \]

\( \psi_0 (r, t) \) is the ground state wave function. The presence of a large number of particles in a ground state permits the introduction of c-number order parameter \[ \psi = \langle a_0 \rangle = \sqrt{N_0} \, e^{i \phi} \]

\( \sqrt{N_0} \) determines the amplitude of the order parameter, \( \phi \) is the phase.

At the phase transition, the phase \( \phi \) is fixed.

This means that the superfluid state is a coherent state. But the number of particles in the condensate can fluctuate.
The operators \( \hat{N}_0 \) and \( \hat{\phi} \) are conjugate operators, like \( \hat{x} \) and \( \hat{p} \). They obey uncertainty relations:

\[
\text{\( \Delta N_0 \Delta \phi \geq \frac{\hbar}{2} \)}
\]

In the BEC, the phase is fixed. Thus, the continuous symmetry related to the phase choice is broken.

\( \text{BEC = broken gauge symmetry} \)

Flow equation: (continuity)

\[
\frac{d}{dt} \psi_s(\vec{r},t) + \text{div} \frac{\nabla}{m} \psi_s(\vec{r},t) = 0
\]

\[
\psi_s(\vec{r},t) = -\frac{i\hbar}{2m} \left( \psi_{0}^* \nabla \psi_{0} - \psi_{0} \nabla \psi_{0}^* \right)
\]

where \( \text{div} = \frac{\partial}{\partial x} (\vec{\psi})_x + \frac{\partial}{\partial y} (\vec{\psi})_y + \frac{\partial}{\partial z} (\vec{\psi})_z \)

and \( \text{gradient} = \hat{\nabla} = \frac{\partial}{\partial x} \hat{e}_x + \frac{\partial}{\partial y} \hat{e}_y + \frac{\partial}{\partial z} \hat{e}_z \).

\( \psi_0(\vec{r},t) = \sqrt{N_0(\vec{r},t)} e^{i\phi(\vec{r},t)} \)

\( \psi_s(\vec{r},t) = \frac{\nabla \varphi(\vec{r},t)}{m} \)

\( \varphi(\vec{r},t) \)
Remember that
\[ \mathbf{j}_s (r^2, t) = \mathbf{q}_s (n, t) \mathbf{v}_s \]
\[ \implies \mathbf{v}_s = \frac{\hbar}{m} \nabla \varphi (r, t) \]
The gradient of the phase determines the superfluid velocity in the system.
Let us quantized the circulation of velocity around the vortex.

\[ \oint \mathbf{v}_s \, d\mathbf{l} = \frac{\hbar}{m} \Delta \varphi \]

where \( \Delta \varphi \) is the total change of phase along the contour \( c \).

\[ \Delta \varphi = 2\pi \hbar, \quad \hbar = 0, 1, \pm 2 \]
The parameter \( n \) should be an integer number to ensure that the order parameter \( \langle \Psi_0 (r) \rangle \) is single valued.
Thus, the circulation is quantized in units $\frac{\hbar}{m}$:

$$\oint_{\gamma} \mathbf{v}_S \, dl = \frac{\hbar}{m} 2\pi n$$

$n = 0, \pm 1, \pm 2, \ldots$ 

This is Onsager - Feynman quantization conditions. Most important vortices have $n = \pm 1$.

The quantization of the vortices is the manifestation of the quantum nature of superfluidity and confirms it’s interpretation as a macroscopic quantum phenomenon.

$$\mathbf{v}_s = \frac{\hbar}{m} \mathbf{\nabla} \varphi (r,t)$$

- The superfluid velocity, is very different from the tangential velocity of rigid bodies $\Rightarrow \mathbf{v} = \mathbf{\omega} \times \mathbf{r}$.

At large distances from the z-axis, the tangential velocity of superfluid approaches to zero, while the tangential velocity of rigid bodies increases.

How we can see it:

$$|\mathbf{v}_s| = \frac{\hbar}{m} |\mathbf{\nabla} \varphi| = \frac{\hbar}{m} r \frac{\partial \varphi}{\partial r}$$ (in cylindrical coordinates)
The energy of quantized vortex

\[ E_V = E(n \neq 0) - E(n = 0) \]

\[ = L \pi \frac{N_0 \hbar^2 n^2}{\mu} \ln \left( \frac{R}{R_c} \right) \]

\( R_c \) is the size of the vortex core.

(\( R_c \) is determined by Cross- Ptievsky equation which is quantum Heisenberg equation of motion for the condensate wave function (\( \psi_0 \)).

\[ \frac{\hbar}{i} \frac{d}{dt} \psi_0(r, t) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(r, t) + g \left| \psi_0(r, t) \right|^2 \right] \psi_0(r, t) \]

\( E_V \) is the energy of the vortex in the laboratory (rest) frame. If a cylinder is rotating with angular velocity \( \omega \) about \( z \)-axis, we must evaluate the energy in rotating reference frame.

\[ E_V' = E_V - \omega \cdot \vec{L}_z \]

where \( \vec{L}_z \) is the angular momentum of the superfluid in the lab. frame.
The ground state solution with no vortices carries $L_z = 0$. When vortices are created, $L_z = j_0 \pi R^2 L \cdot n \hbar$.

It is easy to see that in the rotating reference frame the vortex solution with $\omega L_z > 0$ becomes energetically more preferable compared to $L_z = 0$ solution if $\omega > \omega_c$.

$$\omega_c = \frac{E_v}{L_z} = \frac{\hbar (2n-1)}{MR^2} \ln \left( \frac{R}{r_c} \right)$$

critical angular velocity

Since a superfluid cannot rotate in a rigid way, the rotation is realized through the creation of quantized vortices. Notice that the dependence of $E_v$ on $r_c$ is logarithmic, the energy of the vortex depends weakly on $r_c$. 