Lecture 34

Critical exponents

The critical point is the point at which an order parameter begins to grow continuously from zero:

\[ n \]

\[ T \]

\[ T_c \]

Systems exhibit a new behavior below the critical point, however already approaching the CP from above, the system anticipates its new behavior by adjusting a microscopic state. These adjustments appear in the form of fluctuations which become very large as the CP is approached.

In the vicinity of the critical point many physically different systems show the same behavior. For example, very many of the observed second order phase transitions have a behavior which only depends on the dimension of the system and the symmetry of the order parameter.

This property is called universality.

The critical exponents describe how different thermodynamic quantities behave in the vicinity of the CP.

Systems which belong to the same universality class, have the same critical exponents.
Denote $\epsilon = \frac{T-T_c}{T_c}$ - a reduced distance from the critical point.

Near $T_c$, all thermodynamic functions may be written as

$$f(\epsilon) = A\epsilon^\gamma (1 + B \epsilon^y + \ldots) \quad y > 0$$

The critical exponent for the function $f(\epsilon)$ can be defined as

$$\gamma = \lim_{\epsilon \to 0} \frac{\ln f(\epsilon)}{\ln \epsilon}$$

Six critical exponents are commonly used:

$d, \beta, \gamma, \delta, \eta, \nu$

$$m = m_0 \epsilon^\beta \quad \text{order parameter (magnetization)}$$

$$\chi = \chi_0 \epsilon^{-\gamma} \quad \text{susceptibility}$$

$$C = C_0 \epsilon^{-\delta} \quad \text{specific heat}$$

$$\xi = \xi_0 \epsilon^{-\nu} \quad \text{correlation length}$$

$$\gamma = D H^{1/\delta} \quad \text{equation of state (}\epsilon = 0\text{)}$$

The last equation describes the behavior of the order parameter along the critical isotherm, i.e., at $T = T_c$, when an external field is applied.

Here we should note that these behaviors refer only to the singular part of the thermodynamic quantity.
This means that in the limit $E \to 0$, any thermodynamic quantity can be decomposed into a "regular part," which show no peculiar behavior, and a "singular part" that may be divergent or have divergent derivatives.

Although the critical exponents for a given quantity are believed to be identical from above or below, the prefactors, or "critical amplitudes," are not usually the same.

The last critical exponent defines the behavior of the two-body correlation function

$$\Gamma(\vec{r}) = \langle m(\vec{r})m(0) \rangle - \langle m(\vec{r}) \rangle \langle m(0) \rangle$$

Here $\langle \rangle$ denotes the statistical average.

$\Gamma(\vec{r})$ measures the "persistence of memory" of spatial variations in the order parameter. For translationally invariant systems, the last term may be rewritten as $\langle m(0) \rangle^2$, because $\langle m(0) \rangle = \langle m(\vec{r}) \rangle$.

$$\Gamma(\vec{r}) = e^{-r/\xi}$$

"Ornstein-Zernike form"

$(T > T_c)$

$\xi$ - correlation length

As we said, when $E \to 0$, i.e., when we approach the CP, the fluctuations are correlated over the whole system. Thus $\xi \to \infty$ at $T=T_c$. Then $\Gamma(\vec{r})$ has no characteristic length in the vicinity of $T_c$. 

$\Box$
\[ \Gamma(r) \rightarrow r^{-(d-2+\eta)} e^{-\frac{r}{\xi}}, \text{ where } d = \text{ dimensionality} \]

The significance of the critical exponents lie in their universality.

It has been shown experimentally that different systems whose critical temperatures differ by orders of magnitude, approximately share the same critical exponents. Once again, an universality class of the system is determined by two parameters:

1. The spatial dimension \( d \).
2. The dimension of the order parameter.

The critical exponents may be calculated, at least in the mean field approximations, but often it is not a trivial task. We will compute it for the Ising model.

For a small number of models, critical exponents are known exactly.

Most notable is the 2D Ising square lattice.

(Onsager, 1944) \( d = 0, \beta = \frac{1}{8}, \gamma = \frac{7}{4} \)

\( d = 0 \) corresponds to a logarithmic divergence of the specific heat \( C \): 

\[ C = C_0 \ln \xi + B \]
Only two of the six critical exponents are independent. This is because of "scaling laws".

Let us first give relations between critical exponents, and only after describe the concept of scaling.

Fisher: \( \gamma = \sqrt{(2-\eta)} \)

Rushbrooke: \( d + 2\beta + \gamma = 2 \)

Widom: \( \gamma = \beta (d-1) \)

Josephson: \( \gamma d = 2 - \alpha \)

These scaling relations are a prerequisite for the understanding of finite size scaling which is a basic tool in the analysis of simulational data near the PT.

**The scaling hypothesis**

The idea of scaling underlies all critical exponent calculations.

As we approach the CP, the distance, over which fluctuations are correlated approaches \( \infty \), and all effects of a finite lattice are wiped out.

\( \Rightarrow \) In the vicinity of \( T_c \), when we change distance from the CP (change the temperature), we do not change the form of the free energy, but only its scale.

Scaling is based on the concept of homogeneous functions.
Definition:
A function $F(\lambda x)$ is homogeneous, if for all values of $\lambda$

$$F(\lambda x) = g(\lambda) F(x)$$

One can show that the most general form of $g(\lambda)$

$$g(\lambda) = \lambda^p$$

\[ \Rightarrow \quad F(\lambda x) = \lambda^p F(x) \]

$F(x)$ is said homogeneous function of degree of $p$.

If we have a function of 2 arguments

$$F(\lambda^p x, \lambda^q y) = \lambda F(x, y)$$

If $\lambda = y^{-1/q}$, then

$$F(x, y) = y^{1/q} F\left(\frac{x}{y^{1/q}}, 1\right)$$

Homogeneous function depends on $x$ and $y$ only through the ratio $\frac{x}{y^{1/q}}$.

Now we can apply these ideas to thermodynamic quantities near the critical point.
Wisdom scaling

At the PT, part of the free energy behaves in a "singular" way. This part causes the "singular" behavior of the response functions.

\[ F = F_s + F_{reg} \]

Assume that \( F_s \) is a homogeneous function, and, thus, scales.

Consider magnetic system in the magnetic field. \( F_{reg} \) doesn't change in any significant way when we approach to \( T_c \).

\( F_s \) contains all singular behavior near \( T_c \).

\[ F_s = F_s(\varepsilon, H) \]

\[ \varepsilon = \frac{T - T_c}{T_c} \]

\( F_s \) is hom. function. Thus,

\[ F_s = (\lambda^p \varepsilon, \lambda^q H) = \lambda F_s(\varepsilon, H) \]

Next we find expression of critical exponents in terms of \( p \) and \( q \).

1. \( m(\varepsilon, H=0) \sim \varepsilon^b \)

\[ m = \frac{\partial F}{\partial H} \]

We differentiate \( F_s \) with respect to \( H \)

\[ \lambda^q m(\lambda^p \varepsilon, \lambda^q H) = \lambda m(\varepsilon, H) \]
Next, we assume $\lambda = \epsilon^{-q^p}$ and $H=0$.  

$\Rightarrow \quad \lambda^q \cdot m(\lambda^q \epsilon, 0) = \lambda^q \cdot m(\epsilon, 0)$

$\epsilon^{-q^p} \cdot m(1, 0) = \epsilon^{-q^p} \cdot m(\epsilon, 0)$

$m(\epsilon, 0) = \epsilon^{1-q^p} \cdot m(1, 0)$

$\Rightarrow \beta = \frac{1-q^p}{q^p}$  
we obtain our first relation

2) Now we obtain relation for $\delta$:

$m(0, H) \sim H^{1/\delta}$

$\delta = 0$, $\lambda = H^{-1/q}$

we differentiate $F_S$ with respect to $H$:

$\lambda^q \cdot m(0, \lambda^q H) = \lambda^q \cdot m(0, H)$

$H^{-1} \cdot m(0, (H^{1/q}) H) = H^{-1/q} \cdot m(0, H)$

$m(0, H) = H^{1-q^p} \cdot m(0, 1)$

$\Rightarrow \delta = q^p 1-q^p$  
second relation

3) The magnetic susceptibility is obtained from the thermodynamic relation

$\chi = - \left( \frac{\partial^2 F_S}{\partial H^2} \right)_T \sim \begin{cases} (\epsilon)^{\delta} & T < T_c \\ (\epsilon)^{\delta} & T > T_c \end{cases}$
Differentiate with respect to \( H \), we get

\[ \lambda^{2q} F'(\lambda^p \varepsilon, \lambda^q H) = \lambda \frac{d}{d \lambda} \left( \varepsilon, \lambda H \right) \]

We set \( H = 0 \), \( \lambda = \varepsilon^{-\frac{1}{p}} \Rightarrow \)

\[ F(\varepsilon, 0) = \varepsilon^{\left(1-2q\right)/p} \frac{d}{d \lambda} \left( \varepsilon, 1, 0 \right) \]

\[ \Rightarrow \gamma = \frac{2q-1}{p} \text{ - third relation} \]

4) The heat capacity

\[ C_H = -\lambda \left( \frac{\partial^2 F}{\partial \varepsilon^2} \right) \sim \varepsilon^{-\delta} \]

by differentiating with respect to \( \varepsilon \), we get

\[ \lambda^{2p} C_H(\lambda^p \varepsilon, \lambda^q H) = \lambda C_H(\varepsilon, 1, H) \]

We set \( H = 0 \), \( \lambda = \varepsilon^{-\frac{1}{p}} \Rightarrow \)

\[ C_H(\varepsilon, 0) = \varepsilon^{\left(1-2p\right)/p} C_H(1, 0) \]

\[ \delta = 2 - \frac{1}{p} \text{ - fourth relation} \]

We have obtained a critical exponents \( \delta, \beta, \gamma, \delta \) in terms of 2 parameters \((q, p)\)

5) we can combine relations and obtain

\[ \delta = \beta (\delta - 1) \]

\[ \delta + \beta (\delta + 1) = 2 \]
Wisdom scaling assumption allows us to obtain exact relations between critical exponents. These exact relations can be checked experimentally. They agree well with exp. values for critical exponents.