Attraction Between Like-charged Conducting Spheres

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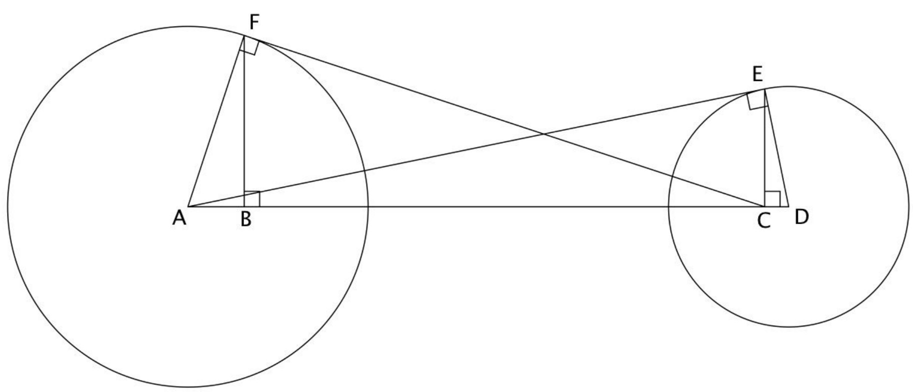

Consider two conducting spheres, which differ in at least one of (radius, potential). The first sphere is centered at A with radius $\overline{AF}$, and is at potential $4\pi\epsilon_0 V_1$; the second sphere is centered at D with radius $\overline{DE}$, and is at potential $4\pi\epsilon_0 V_2$. The task is to find the force between the two spheres. This problem is at least 130 years old; Maxwell knew that if two like-charged spheres are brought close enough together, the force between them is attractive, unless they are identical in both radius and potential. [1]

The problem will be solved using the method of images. Assume $V_2 = 0$; this results in no loss of generality (as will be seen). V_1 may be thought of as due to a charge $q_0 = \overline{AF}\ V_1$, located at A. $V_2 = 0$ is maintained by a charge $p_0 = -q_0 \frac{\overline{CE}}{\overline{AE}}$ located at C, such that $\overline{CD}/\overline{DE} = \overline{DE}/\overline{AD}$. This charge in turn induces an image $q_1 = -p_0 \frac{\overline{BF}}{\overline{CF}}$, placed at B, where $\overline{AB}/\overline{AF} = \overline{AF}/\overline{AC}$, and so on for an infinite series of image charges, all of those within sphere 1 being of one sign, and all within sphere 2 of another. A variety of algebraic relations may be worked out by noting similar triangles.

1 Positions of Image Charges

The positions of the image charges q_n which lie within sphere 1 satisfy

$$x_{n+1} = \frac{(s^2 - R_1^2)x_n - R_2^2}{s^2 x_n - R_2^2}, \tag{1}$$

where R_1, R_2 are the spheres' radii, s is the center-to-center distance, and sx_n is the distance between the nth image and the center of sphere 2. Thus $x_0 = 1$.

Equation 1 is a bilinear transformation (also known as Möbius transformation). Solve for x_n by writing

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \tag{2}$$

where $a = s^2 - R_1^2$, $b = d = -R_2^2$, and $c = s^2$, and then consider the matrix

$$T = \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

a matrix with eigenvalues and eigenvectors

$$\lambda_+ \leftrightarrow \begin{vmatrix} b \\ d - \lambda_- \end{vmatrix}, \lambda_- \leftrightarrow \begin{vmatrix} b \\ d - \lambda_+ \end{vmatrix}, \lambda_\pm = \frac{1}{2}\left((a+d) \pm \sqrt{(a+d)^2 - 4(bc - ad)}\right).$$

Then it may be seen that

$$x_n = \frac{T_{11}^n x_0 + T_{12}^n}{T_{21}^n x_0 + T_{22}^n} = \frac{T_{11}^n + T_{12}^n}{T_{21}^n + T_{22}^n},$$

where T_{ij}^n is element i, j of the matrix T raised to the nth power. A shortcut to evaluating this is as follows: construct the matrices

$$U = \begin{vmatrix} b & b \\ b - \lambda_+ & b - \lambda_- \end{vmatrix}, U^{-1} = \begin{vmatrix} \frac{b-\lambda_-}{\det U} & \frac{-b}{\det U} \\ \frac{-(b-\lambda_+)}{\det U} & \frac{b}{\det U} \end{vmatrix}, \Lambda = \begin{vmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{vmatrix},$$

which satisfy $T = U\Lambda U^{-1}$. Then $T^n = U\Lambda^n U^{-1}$ since $UU^{-1} = I$. This may then be evaluated without too much trouble, so that after some algebra

$$x_n = 1 - \frac{\lambda_+\lambda_-^{n+1} - \lambda_-\lambda_+^{n+1}}{b(\lambda_+^{n+1} - \lambda_-^{n+1}) + \lambda_+\lambda_-^{n+1} - \lambda_-\lambda_+^{n+1}} = 1 - \frac{R_1^2}{\frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+^n - \lambda_-^n} + R_1^2}.$$

Here it is convenient to make the substitution

$$\cosh\alpha = \frac{s^2 - R_1^2 - R_2^2}{2R_1R_2},$$

so that $\lambda_\pm = R_1R_2(\cosh\alpha \pm \sinh\alpha)$. Then the helpful identity

$$(\cosh\alpha \pm \sinh\alpha)^n = \cosh n\alpha \pm \sinh n\alpha$$

can be used to write

$$x_n = \frac{R_2 \sinh{(n+1)\alpha}}{R_2 \sinh{(n+1)\alpha} + R_1 \sinh n\alpha}.$$

This has the limit $x_\infty = R_2^2/(R_2^2 + \lambda_-)$.

2 Magnitudes of Image Charges

The recursive relation between the successive image charges in sphere 1 may be written in two different ways:

$$q_{n+1} = \frac{s^2(1-x_{n+1}) - R_1^2}{R_1 R_2}\, q_n \; ; \quad q_{n+1} = \frac{R_1 R_2}{s^2 x_n - R_2^2}\, q_n ,$$

which allows a second-order difference equation to be written,

$$\frac{1}{q_{n+1}} + \frac{1}{q_{n-1}} = \frac{s^2 - R_1^2 - R_2^2}{R_1 R_2}\frac{1}{q_n} = 2\cosh\alpha\,\frac{1}{q_n}\,. \tag{3}$$

Observe that if $z_n \equiv K e^{n\kappa}$, then the difference equation $z_{n+1} - k z_n + z_{n-1} = 0$ is satisfied if $e^{\kappa} = (k \pm \sqrt{k^2-4})/2$. Comparing this with Eq.(3), we are led to the following assignments:

$$u_n \equiv \frac{1}{q_n}\;, e^{\kappa_+} = \cosh\alpha + \sinh\alpha\;, e^{\kappa_-} = \cosh\alpha - \sinh\alpha\;,$$

and to the general solution for u_n,

$$u_n = A e^{n\kappa_+} + B e^{n\kappa_-} = (A+B)\cosh n\alpha + (A-B)\sinh n\alpha\;,$$

where A and B are determined from the initial conditions, ie the values of q_0 and q_1. Solving for A and B via

$$u_0 = A + B = \frac{1}{q_0}\;, u_1 = \frac{s^2 - R_2^2}{R_1 R_2}\, u_0 = (A+B)\cosh\alpha + (A-B)\sinh\alpha$$

leads to

$$u_n = \frac{u_0}{R_2\sinh\alpha}\left[R_2 \sinh{(n+1)\alpha} + R_1\sinh n\alpha\right]\;,$$

where use has been made of $\sinh{(n+1)\alpha} = \sinh n\alpha\cosh\alpha + \cosh n\alpha \sinh\alpha$. The total charge on sphere 1 is therefore

$$Q_1 = \sum_{n=0}^{\infty} q_n = q_0\; R_2\sinh\alpha \sum_{n=1}^{\infty}\frac{1}{R_2\sinh n\alpha + R_1\sinh{(n-1)\alpha}}\;.$$

In order for sphere 2 to have zero potential, as was assumed at the beginning, it must have a non-zero charge, which is found by summing the values of the image charges $p_n = -q_n R_2/s x_n$. Eliminating $s x_n$ leads to a difference equation for p_n:

$$q_{n+1} = \frac{R_1 R_2 q_n}{s^2 x_n - R_2^2}\;\rightarrow -\frac{1}{p_n} = \frac{R_1}{s q_{n+1}} + \frac{R_2}{s q_n}\;.$$

After some algebra this simplifies to

$$-\frac{1}{p_n} = \frac{1}{q_0}\frac{s\sinh(n+1)\alpha}{R_2\sinh\alpha}\;,$$

which in turn leads to an expression for the total charge on sphere 2,

$$Q_2 = \sum_{n=0}^{\infty} p_n = -\frac{q_0 R_2 \sinh \alpha}{s} \sum_{n=1}^{\infty} \frac{1}{\sinh n\alpha}.$$

Similar expressions for the charges that develop when the potential of sphere 1 is held to zero while the potential of sphere 2 is changed from zero are constructed by exchanging R_1 and R_2.

3 Electrostatic energy

Capacitance is defined by the equation $Q = CV$. In a system with multiple conductors, this becomes a matrix equation:

$$\begin{vmatrix} Q_1 \\ Q_2 \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \end{vmatrix} .$$

The elements of the capacitance matrix are just the charges that develop on the spheres when one of them is held to zero potential and the other is held to a potential of 1, which have been found already. Explicity:

$$\begin{aligned} c_{11} &= 4\pi\epsilon_0 R_1 R_2 \sinh\alpha \sum_{n=1}^{\infty} \frac{1}{R_2 \sinh n\alpha + R_1 \sinh (n-1)\alpha} , \\ c_{12} = c_{21} &= -\frac{4\pi\epsilon_0 R_1 R_2 \sinh\alpha}{s} \sum_{n=1}^{\infty} \frac{1}{\sinh n\alpha} \\ c_{22} &= 4\pi\epsilon_0 R_1 R_2 \sinh\alpha \sum_{n=1}^{\infty} \frac{1}{R_1 \sinh n\alpha + R_2 \sinh (n-1)\alpha} . \end{aligned}$$

Thus far is in Ref. [2]. The capacitance matrix may be inverted, giving the elastance matrix, which is used to find the potentials, if the charges are known:

$$\begin{vmatrix} V_1 \\ V_2 \end{vmatrix} = \begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix} \begin{vmatrix} Q_1 \\ Q_2 \end{vmatrix} ,$$

where the elastance matrix is the inverse of the capacitance matrix,

$$e = \begin{vmatrix} \frac{c_{22}}{\det C} & \frac{-c_{12}}{\det C} \\ \frac{-c_{21}}{\det C} & \frac{c_{11}}{\det C} \end{vmatrix} .$$

The energy of this two-sphere system of conductors (for which $e_{12} = e_{21}$) is

$$E = \frac{1}{2} \sum_i Q_i V_i = \frac{1}{2} \left(e_{11} Q_1^2 + 2 e_{12} Q_1 Q_2 + e_{22} Q_2^2 \right) .$$

If the charges on the spheres have opposite sign, the energy is a monotonically increasing function of the distance between the centers of the spheres, indicating an attractive force at all sphere separations, as expected. However, if the charges on the spheres have the same sign, the energy increases as the spheres are brought together until a maximum is reached, and decreases as the spheres are brought still closer. This indicates that, at some critical separation s_c, the two spheres are in a position of unstable equilibrium, and that if they approach closer than this, the force between them will be attractive.

4 Case $\mathbf{R_1 = R_2}$

If $R_1 = R_2 \equiv R$ the elements of the capacitance matrix can be expressed using Lambert series, $L(x) = \sum_{n=1}^{\infty} x^n/(1-x^n)$:

$$c_{11} = c_{22} = 4\pi\epsilon_0 R \sinh\alpha \frac{2R}{s}\left(L(e^{-\alpha/2}) - 2L(e^{-\alpha}) + L(e^{-2\alpha})\right) , \tag{4}$$

$$c_{12} = c_{21} = -4\pi\epsilon_0 R \sinh\alpha \frac{2R}{s}\left(L(e^{-\alpha}) - L(e^{-2\alpha})\right) \tag{5}$$

where use has been made of the identities [3]

$$\sum_{n=1}^{\infty} \frac{1}{\sinh n\alpha} = 2\left(L(e^{-\alpha}) - L(e^{-2\alpha})\right), \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sinh(n-\frac{1}{2})\alpha} = 2\left(L(e^{-\alpha/2}) - 2L(e^{-\alpha}) + L(e^{-2\alpha})\right) . \tag{7}$$

The location of the critical distance s_c for $R_1 = R_2$ is a function of the charge ratio Q_2/Q_1, as shown in Fig. 1. If the charges on the two spheres are equal, the critical separation $s_c = 2R$, indicating that an attractive force will never be observed. As the charge ratio increases from 1, the critical distance monotonically increases (note: the critical distance for $Q_2/Q_1 < 1$ is not shown in Fig. 1).

This may be qualitatively understood by the following argument. If the charges on the two equal-sized spheres are not equal, then as the spheres are brought closer together, the charge distribution on each sphere will distort. The sphere with larger charge will cause an oppositely-charged area to appear on the nearest point of the sphere with lesser charge, but will not itself develop such an area. As the spheres are brought still closer, the attractive force due to this oppositely-charged portion grows faster than the repulsive force due to the remaining portion of the sphere, because the oppositely-charged portion is closer to the other sphere. If the two spheres have equal charges, then by symmetry the charge densities on the nearest points of the spheres will have the same sign, the above argument fails, and the force remains repulsive.

After the spheres touch, the charges on the two spheres equalize, and the force is therefore repulsive. This sudden change from attractive to repulsive force may be observed using pith balls hanging from strings.

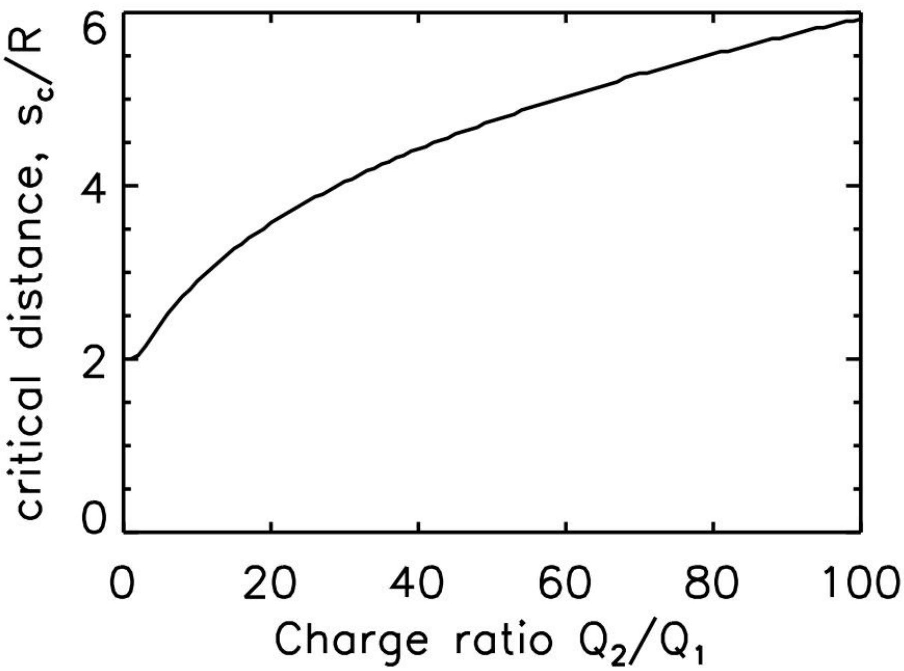

Figure 1: Plot of critical distance for equal-sized spheres, in units of sphere radius, as a function of Q_2/Q_1, for $1 \leq Q_2/Q_1 \leq 100$.

References

[1] J.C. Maxwell, *An Elementary Treatise on Electricity* (Clarendon Press, Oxford, 1888), 2nd. ed., p. 84-86 (§103-104), available online at <http://books.google.com/books?id=C0dWAAAAMAAS&Source=gbs_ViewAPI>

[2] W.R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York, NY, 1950), 2nd ed., pp. 118-121.

[3] J.M. Borwein and P.B. Borwein, *Pi and the AGM* (Wiley-Interscience, New York, 1987), 1st ed., pp 91-92.