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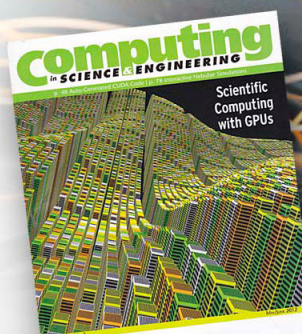
Optimal pacing for running 400- and 800-m track races

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We present a toy model of anaerobic glycolysis that utilizes appropriate physiological and mathematical consideration while remaining useful to the athlete. The toy model produces an optimal pacing strategy for 400-m and 800-m races that is analytically calculated via the Euler-Lagrange equation. The calculation of the optimum v(t) is presented in detail, with an emphasis on intuitive arguments in order to serve as a bridge between the basic techniques presented in undergraduate physics textbooks and the more advanced techniques of control theory. Observed pacing strategies in 400-m and 800-m world-record races are found to be well-fit by the toy model, which allows us to draw a new physiological interpretation for the advantages of common weight-training practices. © *2013 American Association of Physics Teachers.*
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I. INTRODUCTION

The sport called track and field in the USA includes running competitions at distances ranging from 60 m to 10,000 m. Accurate records of finish times go back more than a century for a variety of standard distances and include "world records," in which an athlete ran faster for a given distance than anyone had run before. For many of these world-record performances, accurate measurements were also made of the time it took the record-breaking athlete to reach the halfway point of the race. The subset of world-record performances for standard track races, as ratified by the International Association of Athletics Federations (IAAF), for which halfway "split" times have been published by the IAAF[1] are summarized in Table I. Here, "one-lap" includes the 400 m and the 440-yd imperial equivalent (402.34 m), and "two-lap" includes the 800 m as well as the 880 yd (804.67 m). All world records for the 100 m and 200 m have been set with "negative splits," in which the second half of the race distance is covered faster than the first half. All world records for one-lap races have been set with positive splits (second half slower than first half), and almost all records for two-lap races have been set with positive splits. For longer races, there is no strong preference for positive or negative splits. A natural question to ask is what makes the one-lap and two-lap races different from shorter and longer races?

Recent improvements in video camera technology have increased the number and accuracy of the midway split times available for many world-record performances. Figure 1 shows the velocity profiles $v(t)$ for four current men's world-record performances for which multiple midway split times are available. For each of these distances, the men's world record was set more recently than the women's world record, and split times (see Ref. 1) are more numerous, more accurate, or both. The velocity profile for the 100 m shows that it took the runner at least half the race to reach top speed, followed by only a slight deceleration in the last 20% of the race. The velocity profiles for the mile and 10,000 m show a relatively constant pace through the entire race. However, unlike the other races, the velocity profile for the 400 m shows that maximum speed was reached quite early in race, followed by continuous deceleration. This velocity profile serves as the main empirical datum to be addressed by the toy model herein presented.

II. THEORETICAL UNDERSTANDING OF PACING STRATEGIES OBSERVED IN LONG AND SHORT RACES

There is general agreement that the pacing strategies in short races and in long races—but not in 400 m or 800 m races—can be well understood by simple theoretical models based on Newton's second law (force balance) and/or the first law of thermodynamics (power balance). One representative model from each class is discussed below.

Keller[2,3] derived the optimal race strategy for races up to 10,000 m, including the one-lap and half-lap races, via the calculus of variations, from a model based on Newton's second law. Keller wrote the equation of motion as

$$\frac{dv}{dt} + \frac{v}{\tau} = f(t),\qquad(1)$$

where v is the runner's speed as a function of time t, τ is a constant characterizing the resistance to running, assumed to be proportional to running speed, and $f(t)$ is the propulsive force per unit mass, constrained to be less than a maximum propulsive force F. Empirical knowledge of human exercise physiology is expressed in the assumed relation between propulsive force and energy supply

$$\frac{dE}{dt} = \sigma - fv,\qquad(2)$$

where E represents the runner's energy supply, with finite initial value E_0, that is replenished at constant rate σ. In spite of this replenishment, the energy supply is depleted, reaching zero at the end of the race. The constants τ, σ, E_0, and F are found by comparing the optimal race times to the existing world records for 22 race distances ranging from 50 yd to 10,000 m.

For all race distances, this "energy depletion" model predicts that the second half of the race will be faster than the first. For race distances shorter than a critical distance D_c, estimated to be $D_c = 291$ m, the optimal strategy was found to be for the runner to run at maximum propulsive force for the entire race. The model correctly predicts that runners in 100- and 200-m races will run negative splits and convincingly attributes this to the need to overcome inertia at the start. For distances longer than D_c, the optimal strategy was

Table I. Number of positive and negative splits for the subset of IAAF-ratified world records for which halfway times are recorded by the IAAF.[1]

	Men		Women	
	Positive	Negative	Positive	Negative
100 m	0	6	0	0
200 m	0	6	0	3
One-lap	25	0	9	0
Two-lap	27	2	12	1
Mile	19	12	5	4
3000 m	12	9	0	2
10,000 m	22	16	0	4

to run at maximum propulsive force for up to 2 s at the start, followed by a constant speed for most of the race. This pacing strategy is quite similar to the constant-speed pacing observed in track races with distances of a mile[4] and longer[5] for which the initial acceleration phase lasts for a negligible fraction of the race (see Fig. 1).

Keller's model has the virtue of simplicity and can be extended in many ways. Woodside[6] has modeled longer races, up to 275 km, by including an additional parameter describing fatigue. Behncke[7,8] has proposed a complex model in which Newton's second law is supplemented by a hydraulic model of energy sources in muscular exercise, which itself is based on the model given by Margaria.[9] Regarding races sufficiently long that a runner cannot maintain maximum propulsive force for the whole race, these models support Keller's model in proposing that the best strategy for the athlete is to run with maximum propulsive force for a short time at the beginning of the race and then maintain an approximately constant speed for most of the rest of the race.

Ward-Smith[10] pursued a model based on the first law of thermodynamics and a more detailed knowledge of the bioenergetics of human metabolism. Energy, initially in the form of chemical stored energy, is modeled as coming from two mechanisms: an aerobic mechanism, with power output that rises exponentially to an asymptotic maximum power output R, and an anaerobic mechanism that decays exponentially; the runner always uses the maximum available power. Energy expenditures are modeled as energy degraded to heat during the act of running, energy needed to overcome air drag, and energy converted to kinetic energy of the center-of-mass. The power balance equation $P_{\text{anaer}} + P_{\text{aer}} = P_{\text{heat}} + P_{\text{drag}} + dE_{\text{kin}}/dt$ is then written as

$$\lambda S_0 e^{-\lambda t} + R = Av + \frac{1}{2}\rho v^3 S C_D + mv\frac{dv}{dt} \qquad (3)$$

and is numerically integrated to give $v(t)$. Here, m is the runner's mass, S the runner's effective surface area, and ρ the ambient air density. The remaining five parameters of the model are deduced from comparing the results of the model to finishing times of runners in the Olympic Games, aided by comparison with results of various treadmill tests reported in the literature. This "maximum available power" model does an excellent job predicting in detail the pacing strategy used by runners in the 100 m, including the small but apparently unavoidable deceleration in the final 20 m or so of the race.

Van Ingen Schenau and colleagues subsequently pursued a model based on the first law of thermodynamics,[11–13] which clarified the kinetics of the aerobic and anaerobic pathways of energy production and relaxed the assumption that the runner always uses the maximum available power. This model will be discussed further in Sec. VI.

III. TOY MODEL OF ANAEROBIC METABOLISM

It has long been known that muscle fibers that are forced to work in anaerobic conditions (without a supply of oxygen) quickly cease to function.[14,15] Elite athletes in 400- and 800-m races report an end-of-race sensation that likely sound familiar to average athletes: "burning legs," in which a burning sensation in the working muscles in the legs gets progressively more intense during the race, until at the end of the race, the leg muscles are on the verge of ceasing to function.[16,17] This experience naturally suggests a toy model representing anaerobic metabolism in working muscle, as shown in Fig. 2.
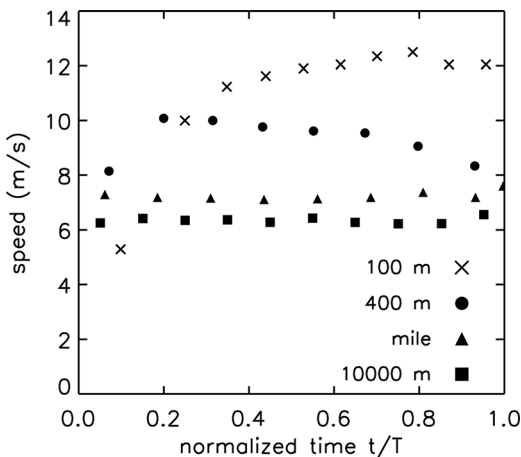

Fig. 1. Observed velocity $v(t)$ for current world record races in the men's 100 m (Usain Bolt, 9.58 s), 400 m (Michael Johnson, 43.18 s), mile (Hicham El Guerrouj, 3:43.13), and 10,000 m (Kenenisa Bekele, 26:17.53).
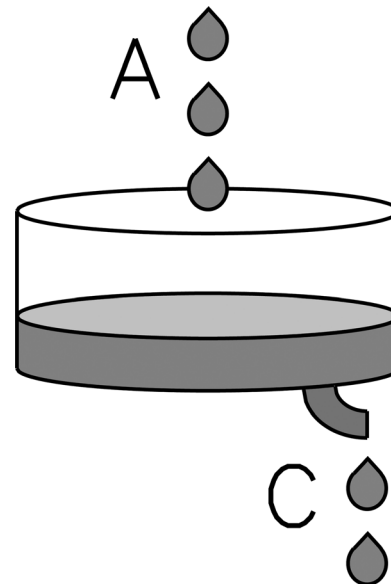

Fig. 2. Physical model representing anaerobic metabolism. Working muscle is conceived as a reservoir for X-factor, added at rate described by A and removed at rate described by C. Muscle ceases to function when reservoir fills.

Process A represents the production rate of X-factor, an as-yet-unspecified metabolite that is associated with the cessation of muscular function when working in anaerobic conditions. Specifically, $A(v)$ is the rate of production of X-factor and is assumed to be a function of running speed v. The quantity X represents the amount of X-factor accumulated in working muscle. When X reaches some particular maximum value in a working muscle, that muscle ceases to function. Process C represents the removal or disappearance of X-factor from working muscle.

The key assumption of the model (see Sec. VII) is that the rate of removal, or disappearance rate, of X-factor is proportional to the amount of X-factor accumulated. With this assumption the amount $X(t)$ of X-factor in working muscle is described by the differential equation

$$\frac{dX(t)}{dt} = A(v) - CX(t), \tag{4}$$

where C is a constant and the running speed v is itself a function of time. Without loss of generality, we suppose the runner starts a race of distance d at time $t=0$, position $s=0$, and with $X=0$, and finishes the race at $t=T$ with $X(T)=1$.

Using this model, a determination of optimal race strategy is perfectly suited to treatment by the calculus of variations, leading to an Euler-Lagrange equation. Rather than consider the problem that presents itself to the runner—how to minimize the time needed to cover a given distance d—it is analytically more convenient to consider the related problem of how to maximize the distance covered in a given time T. This leads to the equation

$$s = \int_{\text{start}}^{\text{finish}} ds = \int_0^T v(t, X, \dot{X}) \, dt, \tag{5}$$

where v is now considered a function of X and $\dot{X} = \partial X/\partial t$ through Eq. (4). Using Eq. (5), the Euler-Lagrange equation is

$$\frac{\partial v}{\partial X} = \frac{d}{dt}\left(\frac{\partial v}{\partial \dot{X}}\right). \tag{6}$$

However, because X and $\dot{X}$ enter only in the combination $\dot{X} + CX \equiv y$, the Euler-Lagrange equation can always, regardless of the functional form of $A(v)$, be written as

$$C\frac{dv}{dy} = \frac{d}{dt}\frac{dv}{dy}, \tag{7}$$

and then integrated once to give

$$\frac{dv}{dy} = ke^{Ct}, \tag{8}$$

where k is a constant of integration. Rewriting Eq. (4) as $y = A(v)$ and inverting gives $v = A^{-1}(y)$, from which the left-hand-side of Eq. (8) follows, and is then rewritten in terms of v by substituting $y = A(v)$. The resulting equation then gives the optimal $v(t)$ in terms of the unknown constant k. To solve for k, we solve Eq. (4) for $A(v)$, multiply both sides by the integrating factor e^{Ct}, and integrate from 0 to T to get

$$X(T)e^{CT} - X(0) = e^{CT} = \int_0^T A\{v(t;k)\}e^{Ct}dt. \tag{9}$$

Figure 2 and Eq. (4) comprise the primary features of the toy model. Two additional secondary features, the physical interpretation of Eq. (4) and the identification of the X-factor, will be treated in Sec. VII. Note that it is the nature of toy models such as this that they are not improved by the addition of detail. This toy model does not attempt a complete accounting of all relevant physiological processes, such as might be made by a model created by a physiologist but attempts instead to call attention to one particular physical principle that (it will be proposed) physiological processes have no choice but to obey.

IV. EXAMPLE

One tires more quickly the faster one runs. This implies that any reasonable $A(v)$ must satisfy $dA(v)/dv > 0$ for all v. As a concrete example consider $A(v) = A_n v^n$, where A_n and $n > 0$ are free parameters. Equation (4) may be rearranged to give

$$v = \left(\frac{\dot{X} + CX}{A_n}\right)^{1/n} = \left(\frac{y}{A_n}\right)^{1/n}, \tag{10}$$

from which the left side of Eq. (8) is

$$\frac{dv}{dy} = \frac{1}{nA_n}\left(\frac{y}{A_n}\right)^{(1-n)/n} = \frac{1}{nA_n}v^{1-n}. \tag{11}$$

Inserting this into Eq. (8), the optimal v is found in terms of the unknown constant k as

$$v = (nA_n k)^{1/(1-n)}e^{-Ct/(n-1)}. \tag{12}$$

We find k by using this velocity in Eq. (9) and integrating. The optimal time histories for v and X are thus found to be

$$v = \left\{\frac{Ce^{CT}}{A_n(n-1)[1 - e^{-CT/(n-1)}]}\right\}^{1/n} e^{-Ct/(n-1)} \tag{13}$$

and

$$X(t) = \frac{e^{CT}}{1 - e^{-CT/(n-1)}}\left[e^{-Ct} - e^{-nCt/(n-1)}\right]. \tag{14}$$

For $n > 1$, the optimal speed as a function of time is monotonically decreasing, leading to positive splits. Only for $n < 1$ is the optimal velocity monotonically increasing with time, giving negative splits.

As will be argued in Sec. VII, a physically reasonable $A(v)$ should satisfy not only $dA/dv > 0$ but also $d^2A/dv^2 > 0$ for all v. The preceding result then implies that for such an $A(v)$, an optimal pacing strategy (assuming one exists) requires positive splits. The race may be imagined as divided into segments, during each of which the speed is restricted to a sufficiently small range so that $A(v)$ over that range may be approximated as a power law, with appropriate $n > 1$. Then, the speed during each segment, and therefore the speed during the whole race, will be a monotonically decreasing function of time.

V. ACCOUNTING FOR THE START

One obvious drawback of the optimal $v(t)$ found above is that the initial velocity $v(0) \neq 0$, which contradicts the fact

that runners start from rest. It might be thought that one way around this would be to suppose that X-factor production also depends on acceleration and write $A = A(v, \dot{v})$. One might suppose that in an increment of time Δt, an increment of chemical stored energy ΔE is converted into an increment of work ΔW and an increment of heat ΔQ, and that this is accompanied by the generation of an increment of X-factor ΔX. Because $\Delta E = \Delta W + \Delta Q$, if one further supposes that both ΔQ and ΔX are proportional to ΔE and that ΔW is to be identified with the change in mechanical energy, then for races on level ground one is led to the *Ansatz*

$$\Delta X \propto \Delta W = \frac{d}{dt}\left(\frac{1}{2}\,mv^2\right)\Delta t = mv\,\dot{v}\,\Delta t \Rightarrow A = A_0 v\,\dot{v}\,. \tag{15}$$

This *Ansatz* is a dead end. Mathematically one can see this by treating the equation $\dot{X} + CX - A(v, \dot{v}) = 0$ as a constraint and using the method of Lagrange multipliers to get

$$s = \int_{\text{start}}^{\text{finish}} ds = \int_0^T \{v + \lambda(t)[\dot{X} + CX - A(v,\dot{v})]\}\,dt\,, \tag{16}$$

which produces two Euler-Lagrange equations. The equation $\partial L/\partial X - d/dt\,[\partial L/\partial \dot{X}] = 0$ leads to $\lambda(t) = ke^{Ct}$. The terms involving X and $\dot{X}$ in the integral then form a total time derivative and therefore do not contribute to the variation of the integrand. However, any term of the form $A = A_1(v)\dot{v}$ may be written as

$$A = A_1(v)\dot{v} + CA_2(v) - CA_2(v), \tag{17}$$

where $A_2(v)$ is chosen so that $d/dv[A_2(v)] = A_1(v)$. Then the term involving $\dot{v}$ will also be part of a total time derivative and will also not contribute the variation. Thus, the model $\dot{X} + CX = A$ possesses a gauge freedom so that replacing $A_0(v) \rightarrow A_0(v) + CA_2(v) + \dot{v}\,d/dv[A_2(v)]$ has no effect on the "equation of motion" that gives the optimal $v(t)$. Inclusion of terms in $A(v)$ of the form $A_1(v)\dot{v}$ need not be investigated.

Treating the finite limit to acceleration as a constraint and retaining $A = A(v)$, it can be proven that the best possible pacing strategy has two phases: an acceleration phase, limited by the runner's inertia and with monotonically increasing velocity, followed by a deceleration phase, limited by X-factor accumulation and with monotonically decreasing velocity.

A graphical outline of the proof is as follows. Consider two trial pacing strategies as shown in Fig. 3(a): "original" pacing (solid line) and "perturbed" pacing (dashed line). The areas under the two curves are the same (the same distance is covered). The function $A(t)$ governs how much X-factor is generated (as opposed to accumulated) in the working muscle. More X-factor is generated by the perturbed strategy than by the original strategy [areas under the curves in Fig. 3(b)] because the positive perturbation to v happens at a higher v than the negative perturbation to v, and $d^2A/dv^2 > 0$ for all v. This might appear to show the perturbed strategy is inferior. However, a key feature of Eq. (4) is that the more X-factor is present, the more can be removed. As may be seen from Eq. (9), $X(T)$ increases as $\int_0^T A(t)e^{Ct}dt$ increases. Therefore, the relative $X(T)$ produced by the two pacing
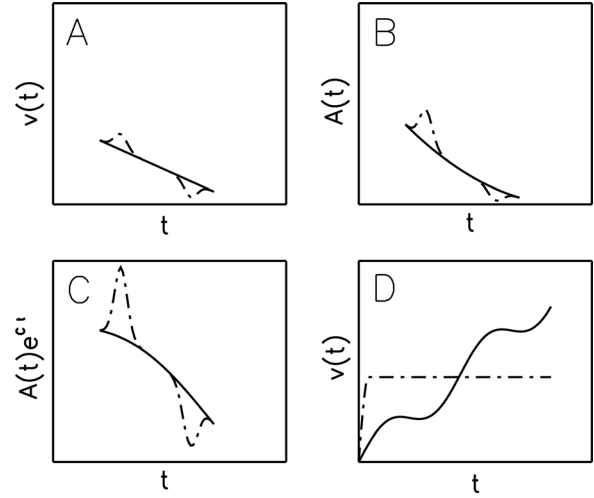

Fig. 3. (a) Two pacing strategies that cover the same distance (areas under the two curves are the same): an "original" strategy (solid line) and a "perturbed" strategy (dotted-dashed line). (b) X-factor generated during the course of the race by the perturbed pacing strategy (area under dashed curve) is more than the X-factor generated by the original pacing strategy (area under solid curve). (c) X-factor accumulated in the working muscle by the end of the race may or may not be larger for the perturbed compared to the original strategy, depending on the details of the perturbation (area under dashed curve may or may not be larger than area under solid curve). (d) A case in which perturbed strategy is superior to original strategy.

strategies may be compared by comparing the areas under the two curves in Fig. 3(c). The perturbed strategy is advantaged because the positive perturbation to v happens early. The superior strategy is the one that results in the lowest $X(T)$; exactly which strategy is superior depends on the details of the perturbation.

Taking the perturbations to be infinitesimal in nature and occurring at times t_1 and $t_1 + \delta$, where δ is finite, the perturbed strategy is better than the original strategy if

$$\frac{\left.\dfrac{dA(v)}{dv}\right|_{t_1}}{\left.\dfrac{dA(v)}{dv}\right|_{t_1+\delta}} < \frac{e^{C(t_1+\delta)}}{e^{Ct_1}} = e^{C\delta}. \tag{18}$$

Using $A(v) = A_n v^n$ as an example one finds that the perturbed and original strategies are equally good if $v(t_1 + \delta) = v(t_1)e^{-C\delta/(n-1)}$, which is the same time dependence found from the Euler-Lagrange equation (13).

Similar reasoning shows that a pacing strategy that does not start out at maximum possible acceleration is inferior to one that does. Figure 3(d) shows an arbitrary pacing strategy (solid line) which is inferior to a pacing strategy that starts at maximum acceleration and then continues at a constant speed (dashed line).

By applying a sequence of infinitesimal perturbations of the type described above, it may be shown that no possible pacing strategy is superior to the following. The runner starts with maximum possible acceleration until such time as the Euler-Lagrange optimum pacing connecting the runner's current (t, X) and the finish $(T, X = 1)$ does not require an instantaneous jump in speed (assumed to be forbidden). The runner then continues at the Euler-Lagrange optimum pacing until the finish. Similar results are well-known in optimal control theory.[18] The best possible pacing strategy is shown

in Fig. 4 along with two perturbations. The perturbation shown by the dashed curve in Fig. 4(a) fails to satisfy Eq. (18) and results in a higher X_F at the finish than the best possible pacing. The perturbation shown in Fig. 4(b) is inadmissible because it exceeds the maximum possible acceleration.

Thus is shown one of the main results of this model, that an athlete trying for a best performance in a 400- or 800-m race should go out as hard as possible at the beginning of the race, running with maximum acceleration, even beyond the time at which the pace is reached that could be held until the end of the race. However, the athlete should consciously reduce power output at a later time and avoid running at maximum available power for too long at the start of the race.

VI. COMPARISON OF MODELS

Attempts were made to reproduce the split times recorded during the men's 400- and 800-m world-record races using Keller's "energy depletion" (ED) model, Ward-Smith's "maximum available power" (MAP) model, and the X-factor accumulation model. In the X-factor accumulation model, the acceleration phase is described as $v(t) = v_m(1 - e^{-c_2 t})$, the same mathematical form used in the ED model. As mentioned, we assume that X-factor accumulates from a value of $X_0 = 0$ (at $t = 0$) to a value of $X_F = 1$ (at $t = T$) and use a function $A(v) = A_0 \tan(\pi v/2 v_m)$. This form for $A(v)$ was preferred to other plausible forms, such as $A(v) = A_n v^n$ or $A(v) = A_0 e^{\alpha v}$, because it introduces fewer free parameters and is consistent with the runner's experience that there is a maximum possible speed beyond which it is not possible to run.

During the men's 400-m world-record race (duration 43.18 s), the athletes started from starting blocks equipped with electronic force sensors. The time elapsed between the start of the race and significant force generation by Michael Johnson was recorded to be 0.150 s.[19] The ED, MAP, and X-factor models were therefore implemented to represent an athlete who remained motionless at the start for 0.150 s and crossed the finish line at 43.18 s.

Split times for the ED and MAP models were computed using the published model parameters to generate pacing strategies, resulting in races finishing in 43.27 s for the ED model and 44.31 s for the MAP model. Split times for the ED and the MAP models were then multiplied by 43.03/43.27 and 43.03/44.31, respectively. Lastly, 0.15 s of motionlessness was added to the beginning. For the X-factor accumulation model $T = 43.03$ s. Writing the units of X-factor as [X], the parameter values were $A_0 = 0.016124 \text{X/s}, v_m = 12.2 \text{m/s}, C = 0.0312 \text{s}^{-1}$, and $c_2 = 0.62 \text{s}^{-1}$. The acceleration

Table II. Comparison of split times in the men's 400-m world record with split times predicted by various models.

Distance (m)	Elapsed time (s)			
	Actual	ED	MAP	X-factor
50	6.14	6.03	5.51	6.08
100	11.10	11.30	10.09	11.03
150	16.10	16.57	14.87	16.07
200	21.22	21.84	19.94	21.21
250	26.42	27.11	25.29	26.47
300	31.66	32.38	30.96	31.86
350	37.18	37.65	36.92	37.42
400	43.18	43.18	43.18	43.18

phase described by these values of v_m and c_2, as quantified by time to 5m and time to 20m, is comparable to the acceleration phase measured in all-out starts by 100-m specialists with personal bests in the range 10.2–10.9s.[20,21]

A comparison of the split times predicted by the three models with the actual split times in the 400-m world-record race is shown in Table II. We see that the X-factor model is closer to each of the actual split times than the other two models. The velocity decrement of approximately 20% of maximum velocity, observed during the latter three-fourths of the race, is typical of race strategies observed during 400-m races by world-class, national-class, and regional-class athletes.[22]

No blocks were used during the men's 800-m world-record race (100.91 s, David Rudisha, 2012). By performing a video analysis of the race, the motionless period at the start was estimated to be 0.2 s and the 200-m split time was determined to be 23.5 s; the 400-m and 600-m split times and finish time are as recorded by the clock in the stadium during the race.

Split times for the ED and MAP models were again computed from published model parameters, resulting in races finishing in 105.95 s (ED model) and 104.38 s (MAP model), and then scaled to the world record finishing time (100.71 s, accounting for 0.2 s of motionlessness at the start) as before. The X-factor model parameters are $A_0 = 0.0069 \text{X/s}$, $C = 0.00862 \text{s}^{-1}$, $v_m = 11.0 \text{m/s}$, and $c_2 = 0.52 \text{s}^{-1}$. As shown in Table III, the X-factor accumulation model predicts each of the split times much better than the other two models. The effect of race tactics may be quantified by comparing the splits of David Rudisha's former world record (2010) of 101.01 s (24.5 s, 48.9 s, 74.59 s, 101.01 s).[23–25] The velocity decrement of approximately 10% during the later stages of these world-record races is again typical of what is observed
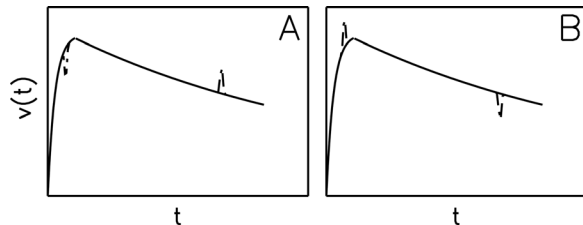

Fig. 4. Two perturbations to the optimal pacing strategy of maximum acceleration followed by optimal deceleration. (a) Perturbations of this type are inferior to the optimal pacing strategy. (b) Perturbations of this type are inadmissible.

Table III. Comparison of split times in the men's 800-m world record with split times predicted by various models.

Distance (m)	Elapsed time (s)			
	Actual	ED	MAP	X-factor
200	23.5	25.60	19.86	24.77
400	49.28	50.59	42.96	49.03
600	74.30	75.58	70.44	74.30
800	100.91	100.91	100.91	100.91

in high-level competition (Gajer et al.,[26] as shown in Fig. 1 of Hanon et al.[27]).

From the point of view of the elite athlete, for both race distances, the three model velocity profiles are distinct. For example, it would not be unreasonable to expect an elite 400-m athlete to be able to go through the 200-m point of a 400-m race within 0.25 s of a preassigned time,[28] whereas the ED and MAP models suggest that the athlete goes through the 200-m point of a 400-m race 0.63 s slower and 1.27 s faster, respectively, than the X-factor model. Graphical comparisons of race data to the predictions of the ED, MAP, and X-factor models, as computed above, are shown in Fig. 5. Figure 5(a) shows the average velocity for each 50-m segment of Michael Johnson's world-record 400-m race (as calculated from the data in the first two columns of Table II), compared to the velocity profiles predicted from the three models. Figure 5(b) shows the average velocity for each 200-m segment of David Rudisha's 2012 world-record 800-m race (as calculated from the data in the first two columns of Table III), compared to the velocity profiles predicted by the three models.

According to the toy model, then, the optimal race strategy for the 400- and 800-m races is somewhere between the ED and MAP predictions. One accelerates at maximum availabl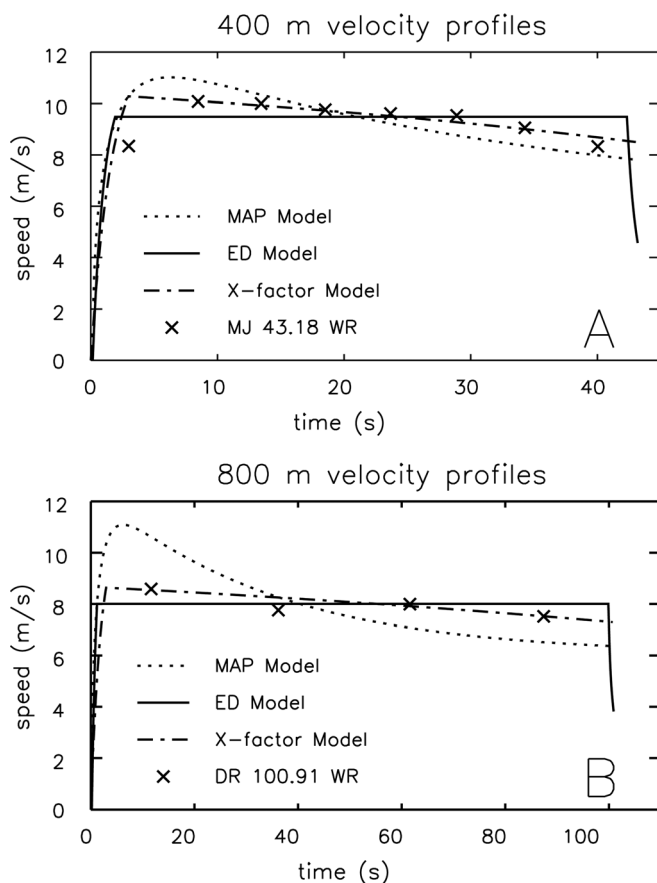e power past the time at which one reaches the velocity one could sustain until the end of the race, and then (at the appropriate time) begins a gradual deceleration, at no time during the deceleration phase running with maximum available power. Numerical investigation of other plausible forms for $A(v)$, such as $A(v) = A_0 v^n$ or $A(v) = A_0 e^{\alpha v}$, suggest this result is robust and does not depend on the specific mathematical functions used for $A(v)$.

A similar result was reached by van Ingen Schenau et al.[11–13] in the context of their model based on the first law of thermodynamics. They observed that the best four athletes in the 1990 world cycling championships 1000- and 4000-m events completed the first parts of their races faster than predicted by a constant-power strategy but slower than predicted by an "all-out" strategy.[12] For the 4000-m race, an optimal pacing strategy was found by trial and error by varying the velocity profile while keeping fixed the parameters describing the athlete and the total energy expended. This strategy was an all-out start lasting somewhat longer than predicted by the constant-power strategy, followed by a gradual decrease in power (amounting to 15% in the case of the 4000-m race, which lasts about 270 s). This was intuitively understood by saying that the kinetic energy the athlete possesses when the finish line is crossed may be thought of as "wasted" kinetic energy.[11] It was concluded that for short-lasting sprint events the optimum strategy is indeed an "all-out" effort, but that the even-pacing strategy should only be used for races lasting longer than 80–100 s.[13] These results are entirely in accordance with the result of the toy model presented here with the amendment that, for running events, the short-duration boundary for the even-pacing strategy is somewhat longer than 100 s.

In summary, the X-factor model, with four free parameters, reproduces observed splits for 400- and 800-m world record races somewhat better than the ED model, which also has four free parameters, and the MAP model, which has five. In praise of the ED and MAP models, it should be mentioned that these models arose from a study of world-record times for a large variety of race distances and therefore they have a wider applicability than just the two distances considered here. The X-factor model prescribes an optimal pacing strategy for 400- and 800-m running races, via the calculus of variations, that is qualitatively similar to the optimal pacing strategy found for the 4000-m cycle race, via trial and error; the latter, however, has the virtue of resting firmly on the results of laboratory tests of athletes.

Fig. 5. Comparison of maximum available power (MAP), energy depletion (ED), and X-factor models to data from world-record races. (A) Velocity profiles predicted by the three models compared to velocity profile of the current men's 400-m world-record race (Michael Johnson, 43.18 s). Each of the eight data points represents the average velocity for a 50-m segment of the race. (b) Velocity profiles compared to the current men's 800-m world-record race (David Rudisha, 100.91 s). Each of the four data points represents the average velocity for a 200-m segment of the race.

VII. DISCUSSION

The traditional view is that muscles cease to function during high-intensity, anaerobic exercise due to a buildup of H^+ ions, which presumably leads to deactivation of enzymes needed for energy production. The strongest support for this comes from the many studies of athletes performing various high-intensity exercises to voluntary exhaustion. For example, Sahlin et al.[29] found via biopsy that the intracellular pH in the working muscle (quadriceps femoris) in exhausted cyclists had dropped from a baseline of 7.0 to 6.4, while Costill et al.[30] found via biopsy that the pH in the gastrocnemius, after an all-out 400-m sprint, had dropped from a baseline of 7.03 to 6.63, and theorized that runners, needing to maintain upright posture, could not tolerate so low a pH as cyclists. Furthermore, ingestion of bicarbonate ("soda loading") improves an athlete's capacity for intense

anaerobic exercise, perhaps due to increased extracellular buffering of H^+.[31]

This traditional view is challenged by a variety of recent research results. For example, the pH of working muscles does not monotonically decrease during anaerobic exercise; upon initiation of exercise, there is a transient increase.[32,33] This transient alkalosis is presumed to result from production of ATP by splitting of phosphocreatine, a process which consumes H^+ ions.[34] An alternative explanation that has been advanced for the cessation of function of the working muscles in 400-m runners is the depletion of phosphocreatine in the working muscles.[35] For a review of literature challenging the traditional view, see Allen *et al.*[36]

Because no single biochemical species may be unequivocally identified as the X-factor, it is suggested that the X-factor is best thought of as entropy. In making this suggestion, no attempt is made to favor one or another of the competing biochemical hypotheses concerning the cause of cessation of muscle function in high-intensity exercise, or to suggest that inadequate heat dissipation is the cause. Rather, the suggestion is made that a muscle at rest may be considered from the point of view of irreversible thermodynamics as a nonequilibrium stationary state, in which entropy-producing processes are sustained at constant levels by continual fluxes of energy and matter and efflux of entropy (see Chapter 1 of Ref. 37 for background on this perspective). From this point of view, the toy model presented here merely expresses for a small part of a living organism (a working muscle) a belief that has long been current among physicists when applied to organisms as a whole, namely, that "the essential thing in metabolism is that the organism succeeds in freeing itself from all the entropy it cannot help producing while alive."[38]

If the X-factor represents entropy, then X represents entropy density and Eq. (4) is to be interpreted as the continuity equation for entropy regarding an entropy-rich region (a working muscle), of entropy density X, enclosed by a boundary that is resistant to the efflux of entropy and surrounded by an entropy-poor region. Equation (4) results from assuming the amount of entropy that crosses the barrier is insufficient to significantly raise the entropy density in the external region. The quantity C describes the rate of diffusive efflux of entropy from the entropy-rich region. The function $A(v)$, describing the rate of generation of entropy as a function of running speed, is expected to grow at least as fast with v as the rate of consumption of energy, since the runner encounters velocity-dependent frictional forces, such as air resistance. Therefore, it should satisfy $d^2A/dv^2 > 0$.

With this interpretation, the difference in race tactics employed in 400-m and 800-m races from the pacing strategies employed in shorter and longer races is understood as follows. Shorter races do not lead to high enough entropy density to cause loss of function in muscles. For longer races, the advantages of starting fast are blunted because the entropy density outside the working muscle (for example, in the blood) can no longer be considered small compared to the entropy density inside the working muscle. Only in the 400-m and 800-m races is the entropy efflux term important enough to recommend positive-split pacing. This interpretation requires no revision of the kinetics of the anaerobic pathway of energy production, as presented by Van Ingen Schenau,[11–13] but merely interprets these kinetics as being the result of a diffusion-limited process.

The X-factor accumulation model suggests that runners training for 400- and 800-m races should perform exercises that maximize the entropy density gradient across the boundaries of the working muscles in the legs. This might be expected to increase the abilities of the working muscles to rid themselves of entropy by diffusion. It seems apparent that the best way to maximize the entropy density gradient is to ask muscles to perform work in the absence of blood flow. It has long been known that sustained muscular contraction reduces blood flow through the contracting muscle,[39] and there seems agreement that cessation of blood flow can be accomplished by sustained contraction at greater than 60%–70% of maximum voluntary force.[40] This is within the range of sustained muscle force required during certain weightlifting exercises, such as squats, favored by sprinters. An athlete might perform an exercise using weight of approximately 70% of the maximum that could be moved in that exercise, and moving the weight slowly, taking on the order of 10 s to perform one up-and-down movement of the weight. At the end of a set of such movements, the athlete has the sensation of "burning" muscles that are on the verge of ceasing to function. It may be speculated that, in addition to building muscular strength, such weightlifting exercises aid performance in the 400- and 800-m races by increasing the abilities of the working muscles in the legs to rid themselves of entropy by diffusion.

VIII. CONCLUSIONS

Runners in 400- and 800-m races are often urged to attempt to run the second half of the race at the same pace as the first (even splits) on the grounds that this is the most economical distribution of energy, as no doubt it is. However, almost no successful runners in 400- and 800-m races actually do this, and the burden of this paper has been to try to understand why. A simple model of entropy accumulation suggests that for any reasonable relation between entropy generation and speed, the optimal race strategy is to run the first half of the race faster than the second. Such a strategy causes the entropy density gradient across the boundary of the working muscle to take on a large value early in the race, which increases the entropy efflux. That this is in fact the optimal race strategy is proven by the overwhelming preponderance of positive-split races in the set of world records for one-lap and two-lap races. While there are many other factors that athletes encounter in races, such as the race tactics of other athletes, as well as psychological variables, taking into account entropy arguments suggests that even-split pacing is not optimal, even in principle, for 400- and 800-m races.

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[1]*Progression of IAAF World Records, 2011 Edition*, edited by Imre Matrahazi (International Association of Athletics Federations, Monaco, 2011).

[2]J. B. Keller, "A theory of competitive running," Phys. Today **26**(9), 42–47 (1973).

[3]J. B. Keller, "Optimal velocity in a race," Am. Math Monthly **81**(5), 474–480 (1974).

[4]T. D. Noakes, M. I. Lambert, and R. Hauman, "Which lap is the slowest? An analysis of 32 world mile record performances," Br. J. Sports Med. **43**, 760–764 (2009).

[5]R. Tucker, M. I. Lambert, and T. D. Noakes, "An analysis of pacing strategies during men's world record performances in track athletics," Int. J. Sports Physiol. Perform. **1**, 233–245 (2006).

[6]W. Woodside, "The optimal strategy for running a race (a mathematical model for world records from 50 m to 275 km)," Mathl. Comput. Modelling **15**, 1–12 (1991).

[7]H. Behncke, "A mathematical model for the force and energetics in competitive running," J. Math. Biol. **31**, 853–878 (1993).

[8]H. Behncke, "Optimization models for the force and energy in competitive running," J. Math. Biol. **35**, 375–390 (1997).

[9]R. Margaria, *Biomechanics and Energetics of Muscular Exercise* (Clarendon Press, Oxford, 1976), pp. 53–55.

[10]A. J. Ward-Smith, "A mathematical theory of running, based on the first law of thermodynamics, and its application to the performance of world-class athletes," J. Biomech. **18**(5), 337–349 (1985).

[11]G. J. van Ingen Schenau, R. Jacobs, and J. J. de Koning, "Can cycle power predict sprint running performance?," Eur. J. Appl. Physiol. **63**, 255–260 (1991).

[12]G. J. van Ingen Schenau, J. J. de Koning, and G. de Groot, "The distribution of anaerobic energy in 1000 and 4000 metre cycling bouts," Int. J. Sports Med. **13**, 447–451 (1992).

[13]G. J. van Ingen Schenau, J. J. de Koning, and G. de Groot, "Optimisation of sprinting performance in running, cycling and speed skating," Sports Med. **17**(4), 259–275 (1994).

[14]O. Meyerhof, "Zur Verbrennung der Milchsäure in der Erholungsperiode des Muskels," Pflügers Arch. Ges. Physiol. **175**, 88–93 (1919).

[15]A. V. Hill and H. Lupton, "Muscular exercise, lactic acid, and the supply and utilization of oxygen," Q. J. Med. **16**, 135–171 (1923).

[16]400 m: Andrew Rock, 400-m US Champion 2006 (private communication).

[17]800 m: Suzy Favor Hamilton, 800-m US Olympian 1996 (private communication).

[18]A. Sage, *Optimum Systems Control* (Prentice-Hall, Englewood Cliffs, NJ, 1968).

[19]Results of the 7th World Championships in Athletics, Seville, 1999 (day 6), <http://www2.iaaf.org/sevilla99/results>.

[20]W. Baumann, "Kinematic and dynamic characteristics of the sprint start," in *Biomechanics VB (Int. Ser. Biomech., vol. Vb)* (University Park Press, Baltimore, MD 1976), pp. 194–199.

[21]M. Čoh, B. Jošt, B. Škof, K. Tomažin, and A. Dolenec, "Kinematic and kinetic parameters of the sprint start and start acceleration model of top sprinters," Gymnica **28**, 33–42 (1998).

[22]C. Hanon and B. Gajer, "Velocity and stride parameters of world-class 400-meter athletes compared with less experienced runners," J. Strength Cond. Res. **23**(2), 524–531 (2009).

[23]Reaction time of 0.3 s and 200-m split determined by the author from analysis of race video.

[24]400-m split was given by Track and Field News as 48.9 s, in "Lap times in 800 m world records since the first sub-1:50," available online at <http://www.trackandfieldnews.com/index.php/display-article?arId=48269>.

[25]600-m split time and finish time are from Ref. 1.

[26]B. Gajer, C. Hanon, J. Marajo, and J.-C. Vollmer, *Le 800 mètres: Analyse descriptive et entraînement* (Institut National des Sports et de l'éducation Physique, Paris, 2000).

[27]C. Hanon, C. Thomas, J.-M. Le Chavalier, B. Gajer, and H. Vandewalle, "How does VO_2 evolve during the 800m?," New Stud. Athletics **17**(2), 61–68 (2002).

[28]Andrew Rock (private communication).

[29]K. Sahlin, A. Alvestrand, R. Brandt, and E. Hultman, "Intracellular pH and bicarbonate concentration in human muscle during recovery from exercise," J. Appl. Physiol. **45**(3), 474–480 (1978).

[30]D. L. Costill, A. Barnett, R. Sharp, W. J. Fink, and A. Katz, "Leg muscle pH following sprint running," Med. Sci. Sports Exerc. **15**(4), 325–329 (1983).

[31]D. Wilkes, N. Gledhill, and R. Smyth, "Effect of acute induced metabolic alkalosis on 800-m racing time," Med. Sci. Sports Exerc. **15**(4), 277–280 (1983).

[32]G. J. Kemp, M. Roussel, D. Bendahan, Y. Le Fur, and P. J. Cozzone, "Interrelations of ATP synthesis and proton handling in ischaemically exercising human forearm muscle studied by ^{31}P magnetic resonance spectroscopy," J. Physiol. **535.3**, 901–928 (2001).

[33]G. Walter, K. Vandenborne, M. Elliott, and J. S. Leigh, "*In vivo* ATP synthesis rates in single human muscles during high intensity exercise," J. Physiol. **519.3**, 901–910 (1999).

[34]E. Hultman and K. Sahlin, "Acid-base balance during exercise," Exerc. Sport Sci. Rev. **8**, 41–128 (1980).

[35]J. Hirvonen, A. Nummela, H. Rusko, S. Rehunen, and M. Härkönen, "Fatigue and changes of ATP, creatine phosphate, and lactate during the 400-m sprint," Can. J. Sport Sci. **17**(2), 141–144 (1992).

[36]D. G. Allen, G. D. Lamb, and H. Westerblad, "Skeletal muscle fatigue: Cellular mechanisms," Physiol. Rev. **88**, 287–332 (2008).

[37]W. Yourgrau, A. van der Merwe, and G. Raw, *Treatise on Irreversible and Statistical Thermophysics* (Dover, NY, 1982), pp. 1–54.

[38]E. Schrödinger, *What is Life?* (Cambridge U.P., Cambridge, 1992), Chapter 6, p. 71.

[39]H. Barcroft and J. L. E. Millen, "The blood flow through muscle during sustained contraction," J. Physiol. **97**(1), 17–31 (1939).

[40]F. A. Gaffney, G. Sjøgaard, and B. Saltin, "Cardiovascular and metabolic responses to static contraction in man," Acta Physiol. Scand. **138**(3), 249–258 (1990).

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