Cosine Potential

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We wish to solve the equation

$$-\frac{1}{2}y''(x) + V_0Cos(x)y(x) = Ey(x) \tag{1}$$

subject to the boundary condition

$$y(0) = y(2\pi) \tag{2}$$

We then have $-y''(x) = 2(E - V_0Cos(x))y(x)$ Let $x = 2q$. Then, $y''(q) + (-8V_0Cos(2q) + 8E)y(q) = 0 \implies y(x) = a_0C(8E, 4V_0, \frac{x}{2}) + a_1S(8E, 4V_0, \frac{x}{2})$, where C(a,q,v) is the Mathieu cosine function, S(a,q,v) is the Mathieu sine function.

For symmetric eigenfunctions, we can take $a_1 = 0$, and normalize ψ based on $\int_0^{2\pi} psi(x)^2\, dx. = 1$.
For asymmeric eigenfunctions, we can take $a_0 = 0$, and normalize in the same manner.
The energies are found by noticing that $\frac{a_r}{8} = E$, where a_r is the Mathieu characteristic for the Mathieu cosine function (i.e. the value of $8E$ needed for the Mathieu cosine function to be periodic), for even functions, and $\frac{b_r}{8}$ for odd functions, with b_r the characteristic for Mathieu sine functions.