Week 3 Lectures

References: Text: Hecht, "Optics," 4th Ed., Sects. 5.2, 5.4, 5.5, 6.7, 8.4, 8.11, 9.4, 9.5

Figure 5.3. Explain. Referring to Fig. 5.3(c):

\[ n_1 (\overrightarrow{FA}) + n_2 (\overrightarrow{AD}) = \text{constant} \]

\[ \Rightarrow (\overrightarrow{FA}) + (\frac{n_2}{n_1})(\overrightarrow{AD}) = \text{constant} \quad \text{HYPERBOCA} \]

Snell's Law:

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \]

Typical Results, Hyperbolic Lenses:

Spherical Sector Interface:

\[ R = \text{radius of curvature} \]
\[ C = \text{center of sphere} \]
\[ V = \text{vertex of surface} \]

Optical Path Length = \( n_1 l_0 + n_2 l_1 \) : OPL

\[ l_0 = \left[ R^2 + (s_0 + R)^2 - 2R(s_0 + R)\cos \phi \right]^{\frac{1}{2}} \]

\[ l_1 = \left[ R^2 + (s_i - R)^2 + 2R(s_i - R)\cos \phi \right]^{\frac{1}{2}} \]
\[ OPL = n_1 \left[ R^2 + (S_o + R)^2 - 2R (S_o + R) \cos \phi \right]^{1/2} + \]
\[ n_z \left[ R^2 + (S_i - R)^2 + 2R(S_i - R) \cos \phi \right]^{1/2} \]

\[ \frac{\partial [OPL]}{\partial \phi} = 0 \quad \text{[Fermat's Principle]} \]

\[ \Rightarrow \quad \frac{n_1 R (S_o + R) \sin \phi}{2 \lambda_0} - \frac{n_z R (S_i - R) \sin \phi}{2 \lambda_1} = 0 \]

\[ \Rightarrow \quad \frac{n_1}{\lambda_0} + \frac{n_z}{\lambda_1} = \frac{1}{R} \left[ \frac{n_z S_i}{\lambda_1} - \frac{n_1 S_o}{\lambda_0} \right] \]

**Paraxial Approximation:**
\[ \cos \phi \approx 1 \quad \sin \phi \approx \phi \quad \lambda_0 \approx S_o \]
\[ \lambda_1 \approx S_i \]

\[ \Rightarrow \quad \frac{n_1}{S_o} + \frac{n_z}{S_i} = \frac{n_z - n_1}{R} \]

**Object Focal Length** \( f_o = \left( \frac{n_1}{n_z - n_1} \right) R \)

**Image Focal Length** \( f_i = \left( \frac{n_z}{n_z - n_1} \right) R \)

**Sign Convention**

\( S_o, f_o \) + Left of \( V \)
\( S_i, f_i \) + Right of \( V \)
\( R \) + If \( C \) is Right of \( V \)

**Virtual Image:** Rays diverge from it

**Virtual Object:** Rays converge toward it
THIN LENS: \( d = \text{LENS THICKNESS} \ll \frac{2}{\text{NA}^2} \)

\( \lambda = \text{WAVELENGTH OF LIGHT} \)

\( \text{NA} = \text{NUMERICAL APETURE} = \eta_d \sin \theta_{\text{MAX}} \)

PARAXIAL ANALYSIS OF THIN LENS:

\[ \frac{n_m}{s_{o1}} + \frac{n_d}{s_{d1}} = \frac{n_d - n_m}{R_1} \]

And surface: RAYS IMPINGE FROM \( P' \), A DISTANCE \( s_{o2} \) FROM 2ND SURFACE, RAYS ARRIVING ARE IN MEDIUM OF INDEX \( n_d \).

\( |s_{o2}| = |s_{d1}| + d \)

\( s_{o2} = |s_{o2}| \quad \text{BUT} \quad s_{d1} = -|s_{d1}| \)

\( \Rightarrow s_{o2} = -s_{d1} + d \)

\[ \Rightarrow \frac{n_d}{(-s_{d1} + d)} + \frac{n_m}{s_{d2}} = \frac{n_m - n_d}{R_2} \]

NOTE: \((n_m-n_d)<0\) AND \(R_2<0\)
\[
\frac{n_m}{S_{o1}} + \frac{n_m}{S_{2z}} = (n_k - n_m) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) + \frac{n_k d}{(S_{z1} - d) S_{z1}}
\]

if \( d \to 0 \) in air:

\[
\frac{1}{S_0} + \frac{1}{S_i} = (n_k - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \quad \begin{cases} \frac{S_{o1}}{S_0} = 1 \\ \frac{S_{i2}}{S_i} = 1 \end{cases}
\]

for a thin lens: \( f_i = f_o (= f) \)

\[
\frac{1}{f} = \frac{1}{S_0} + \frac{1}{S_i} = (n_k - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]

example:

planar-convex lens, radius of curvature 50 mm,
index 1.5

light enters planar surface \( \Rightarrow R_1 = \infty, R_2 = -50 \text{ mm} \)

\[ \Rightarrow \frac{1}{f} = \left( \frac{0,5}{50} \right) \Rightarrow f = 100 \text{ mm} \]

light enters convex surface \( \Rightarrow R_1 = +50 \text{ mm}, R_2 = \infty \)

\[ \Rightarrow \frac{1}{f} = \frac{0,5}{50} \Rightarrow f = 100 \text{ mm} \]

fig. 5.15 (a) - (f), explain. hecht, p. 159

\[
\frac{1}{f} = \left( \frac{n_k - n_m}{m} - 1 \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]

(a) (b) \( n_k > n_m; R_1 > 0, R_2 < 0 \Rightarrow \text{each focal length positive} \)

(c) \( n_k < n_m \Rightarrow f \text{ negative} \)

(d) \( n_k > n_m \text{ but } R_1 < 0 \text{ and } R_2 > 0 \Rightarrow f \text{ negative} \)

object (d) and image (e) are virtual

(f) \( n_k < n_m \Rightarrow f \text{ positive} \)
<table>
<thead>
<tr>
<th>SIGN CONVENTION</th>
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<th>-</th>
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<tbody>
<tr>
<td>$S_o$</td>
<td>REAL OBJECT</td>
<td>VIRTUAL OBJECT</td>
</tr>
<tr>
<td>$S_i$</td>
<td>REAL IMAGE</td>
<td>VIRTUAL IMAGE</td>
</tr>
<tr>
<td>$f$</td>
<td>CONVERGING LENS</td>
<td>DIVERGING LENS</td>
</tr>
<tr>
<td>$y_o$</td>
<td>ERECT OBJECT</td>
<td>INVERTED OBJECT</td>
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<td>$x_o$</td>
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<tr>
<td>$y_i$</td>
<td>ERECT IMAGE</td>
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<td>$x_i$</td>
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<td>$+$: RIGHT OF $F_o$</td>
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$AOF; \text{ AND } P_2 P_1 F; \text{ ARE SIMILAR } \Rightarrow \frac{y_o}{y_i} = \frac{f}{(s_i - f)}$

$S_2 S_1 O \text{ AND } P_2 P_1 O \text{ ARE SIMILAR } \Rightarrow \frac{y_o}{y_i} = \frac{S_o}{S_i} = \frac{f}{(s_i - f)}$

$S_2 S_1 F \text{ AND } B OF_o \text{ ARE SIMILAR } \Rightarrow \frac{f}{(s_o - f)} = \frac{y_i}{y_o}$

$= \Rightarrow x_o x_i = f^2$

TRANVERSE MAGNIFICATION: $M_T = \frac{y_i}{y_o} = \frac{-s_i}{S_o}$
Fig. 5.23 (d), explain.

Longitudinal magnification: \( M_L = \frac{dX_i}{dX_o} = -\frac{f^2}{x_0^2} = -M_T \)

So: \( M_L < 0 \) (inverted along optic axis).

Next generalize to two positive lenses \((L_1, L_2)\) separated by a distance \(d\) less than either focal length.

1. Start by ignoring \(L_2\), construct image formed by \(L_1\), using rays 2 and 3.

2. \(L_1\) constructs ray 4: from \(P_1\) through \(O_2\).

3. Change ray 3, \(L_2\) refracts ray 3 through the image focus \(F_{i2}\).

4. The image of the two lenses together is the intersections of rays 3 and 4.
So when \(|d| < |f_1|, |f_2|, |f_1| + |f_2|\):

If \(f_2 > 0\), \(L_2\) Adds Convergence
If \(f_2 < 0\), \(L_2\) Adds Divergence

Consider two thin lenses separated by a distance greater than the sum of their focal lengths.

(1) Again, rays 2 and 3 fix the position of the intermediate image generated by \(L_1\) alone.

For \(L_1\):
\[
\frac{1}{s_{21}} = \frac{1}{f_1} - \frac{1}{s_{01}} \Rightarrow s_{21} = \frac{s_{01} f_1}{s_{01} - f_1} > 0
\]

For \(L_2\):
\[
s_{02} = d - s_{21}
\]
\(d > s_{21} \Rightarrow s_{02} > 0 \Rightarrow\) Object for \(L_2\) is Real
\[
\frac{1}{s_{i2}} = \frac{1}{f_2} - \frac{1}{s_{02}} \Rightarrow s_{i2} = \frac{s_{02} f_2}{s_{02} - f_2} = \frac{f_2 (d - s_{i1})}{s_{02} - f_2} (d - s_{i1} - f_2)
\]
\[
= \frac{f_2 d - \left[ f_2 s_{01} f_1 / (s_{01} - f_1) \right]}{d - f_2 - \left[ f_2 s_{01} f_1 / (s_{01} - f_1) \right]}
\]

Here \(s_{01}\) and \(s_{i2}\) are the object and image distances of the compound lens.
EXAMPLE:

PLACE OBJECT 50.0 cm FROM 1ST OF 2 LENSES. SEPARATE LENSES BY 20.0 cm. HAVE f_1 = 30.0 cm AND f_2 = 50.0 cm.

\[ z_1 = \frac{1}{s_1} = 2.2 \text{ cm}. \]

WITH REAL IMAGE.

TOTAL TRANSVERSE MAGNIFICATION \( M_T = (M_{T1})(M_{T2}) \)

\[ = \frac{f_1}{d(s_0 - f_1)} \cdot \frac{s_1}{s_0} = -0.72 \]

BACK FOCAL LENGTH: DISTANCE FROM LAST SURFACE OF OPTICAL SYSTEM TO THE 2ND FOCAL POINT OF SYSTEM.

FRONT FOCAL LENGTH: DISTANCE FROM VERTICE OF FIRST SURFACE TO THE FIRST OR OBJECT FOCUS.

\[ d \]

IF \( s_1 \rightarrow \infty \), \( s_0 \rightarrow f_2 \)

\[ s_0 \rightarrow d - f_2 \]

\[ \frac{1}{s_0} \left| \begin{array}{c} 1 \end{array} \right| = \frac{1}{f_1} - \frac{1}{d - f_2} = \frac{d - (f_1 + f_2)}{f_1 (d - f_2)} \]

SIMILARLY: IF \( s_0 \rightarrow \infty \), \( (s_0 - f_1) \rightarrow s_0 \)

\[ B.F.L. = s_1 = \frac{f_2}{d - f_1} \]

EXAMPLE: \( d = 10.0 \text{ cm}, f_1 = -30.0 \text{ cm}, f_2 = 20 \text{ cm} \)

\[ \Rightarrow \text{B.F.L.} = 40 \text{ cm}. \]

\[ \text{F.F.L.} = 15 \text{ cm} \]

\[ \frac{L_1}{L_2} \]

\[ \rightarrow \text{B.F.L.} \]
QED per Richard Feynman

1. Light is made of particles

2. Probability amplitude of an event occurring = \sum \text{constituent probability amplitudes}

All ways event can occur

3. Probability amplitudes are (usually) complex numbers

Snell's law derived differently:

\[ p_i \sin \theta_i \]

\[ \text{INTERFACE} \]

When translational invariance holds, linear momentum is conserved.

\[ p_i \sin \theta_i = p_f \sin \theta_f \]

\[ p_i = \frac{h}{\lambda_i} \quad p_f = \frac{h}{\lambda_f} \]

\[ \Rightarrow \frac{h \sin \theta_i}{\lambda_i} = \frac{h \sin \theta_f}{\lambda_f} \quad \text{Multiply both sides by} \ (\frac{c}{v}) \]

\[ \Rightarrow \frac{n_i \sin \theta_i}{\lambda_i} = \frac{n_f \sin \theta_f}{\lambda_f} \]

QED and a lens

Constant phase, hard to describe clearly.
Focal Planes

Make \( \overline{AC} \parallel \overline{BC}_2 \).

\( \overline{AB} \) enters and leaves lens in same direction.

\( \triangle \overline{AOC}_1 \) and \( \triangle \overline{BOC}_2 \) are similar

\( \Rightarrow \frac{\overline{IR}_1}{\overline{OC}_2} = \frac{\overline{IR}_2}{\overline{OC}_1} \)

\( \overline{IR}_1, \overline{IR}_2 \) are constant

\( \Rightarrow \) location of \( O \) is independent of \( A \) and \( B \)

Show Fig. 5.17 (Hecht, p. 160).

\( \Rightarrow \) bundles focused on spherical sector \( \Sigma \) centered on \( C \), approximate \( \Sigma \) as a plane.

Front focal plane: plane \( \perp \) symmetry axis and containing image focus \( F_0 \),

Back focal plane: contains image focus \( F_1 \),

Thick Lenses

Primary principal plane

Secondary principal plane
NODAL POINTS:

If lens is surrounded on both sides by the same medium, \( n_1 = n_2 \) and \( n_1 = n_2 = n \).

A thick lens can be equated to two spherical refracting surfaces separated by a distance \( d_L \). For a thick lens immersed in air:

\[
\frac{1}{S_0} + \frac{1}{S_i} = \frac{1}{f} \quad \text{if \( S_0 \) and \( S_i \) are measured from the 1st and 2nd principal planes, and \( f \) is:}
\]

\[
\frac{1}{f} = (n \varepsilon - 1) \left[ \frac{1}{R_1} - \frac{1}{R_2} + \frac{(n \varepsilon - 1) d_L}{n \varepsilon R_1 R_2} \right]
\]

Principal planes are at:

\[
\overrightarrow{V_1 H_1} = h_1 \quad \text{and} \quad \overrightarrow{V_2 H_2} = h_2
\]

These are positive when the plane lies to the right of its respective vertex.

\[
h_1 = -\frac{f (n \varepsilon - 1) d_L}{R_1 n \varepsilon} \quad \text{and} \quad h_2 = -\frac{f (n \varepsilon - 1) d_L}{R_1 n \varepsilon}
\]

Also:

\[
x_0 \times i = f^2
\]

And \( M_1 = \frac{y_i}{y_0} = -\frac{x_i}{x_0} = -\frac{f}{x_0} \)
RAY TRACING

1) REFRACTION EQUATION

\[ n_{e1} \alpha_{e1} = n_{i1} \alpha_{i1} - \delta_1 y_i \]

\[ \delta_1 = \frac{(n_{e1} - n_{i1})}{R_1} \]  

SHOW FIG. 6.7,

HECHT, p. 247

2) TRANSFER EQUATION

\[ y_2 = y_1 + d_{21} \alpha_{e1} \]

MATRIX ANALYSIS OF LENSES

\[ n_{e1} \alpha_{e1} = n_{i1} \alpha_{i1} - \delta_1 y_i \]

\[ y_{t1} = 0 + y_i \]

\[ \begin{bmatrix} n_{e1} \alpha_{e1} \\ y_{e1} \end{bmatrix} = \begin{bmatrix} 1 & -\delta_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n_{i1} \alpha_{i1} \\ y_i \end{bmatrix} \]

OR

\[ \begin{bmatrix} n_{e1} y_{e1} \\ y_{e1} \end{bmatrix} = \begin{bmatrix} n_{i1}/n_{e1} & -\delta_1/n_{e1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{i1} \\ y_i \end{bmatrix} \]

OR

\[ \begin{bmatrix} \alpha_{e1} \\ y_{e1} \end{bmatrix} = \begin{bmatrix} \alpha_{i1}/n_{e1} \\ y_i \end{bmatrix} \]

\[ \uparrow \]

REFRACTION MATRIX = \[ R_1 \]

\[ \uparrow \]

TRANSFER MATRIX = \[ T_{21} \]

\[ \begin{bmatrix} n_{i2} \alpha_{i2} \\ y_{i2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d_{21}/n_{e1} & 1 \end{bmatrix} \begin{bmatrix} n_{e1} \alpha_{e1} \\ y_{e1} \end{bmatrix} \]

\[ \uparrow \]

TRANSFER MATRIX = \[ T_{21} \]

\[ \Rightarrow \]

INCIDENT RAY AT \[ P_2 = Y_{c2} = T_{21} R_1 Y_{c1} \]

HECHT, p. 248
Now let me switch nomenclature slightly and use that of

(1) **TRANSLATION MATRIX**

\[
y_2 = RP = RQ + QP = y_1 + t \theta_1; \quad t = SQ = \text{TRANSLATION THROUGH DISTANCE} (t)
\]

This is true whether \( \theta_1 \) is > 0 or < 0.

Let \( n = \text{INDEX OF REFRACTION BETWEEN } RP_1 \text{ AND } RP_2 \).

\[
y_2 = y_1 + \left( \frac{t}{n} \right) (n \theta_1) = y_1 + T(\theta_1, n)
\]

\[
T = \frac{t}{n}
\]

\[
\begin{bmatrix}
y_2 \\
n \theta_2
\end{bmatrix} =
\begin{bmatrix}
1 & T \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
y_1 \\
n \theta_1
\end{bmatrix}
\]

If a region is divided into two parts \( T_1 \) and \( T_2 \), with

same index \( (n) \): \( T_1 = \begin{bmatrix} 1 & T_1 \\ 0 & 1 \end{bmatrix} \) and \( T_2 = \begin{bmatrix} 1 & T_2 \\ 0 & 1 \end{bmatrix} \)

\( G = \text{GEOMETRY OF IMAGES WILL BE THE SAME IF TWO}

(or more) regions are interchanged.

(2) **REFRACTION MATRIX**

\[ n_2 > n_1 \]
The separation between $R_1$ and $R_2 = R (1 - \cos \alpha) \approx 0$ since $\alpha$ is small. $\Rightarrow y_2 = y_1$.

$n_1 \sin \theta_1 = n_2 \sin \theta_2 \Rightarrow n_1 \theta_1 = n_2 \theta_2$

$\theta_1 = \theta_1 + \frac{y_1}{R} \Rightarrow \theta_2 = \theta_2 + \frac{y_1}{R}$

$\Rightarrow n_1 (\theta_1 + \frac{y_1}{R}) = n_2 (\theta_2 + \frac{y_1}{R})$

$\Rightarrow n_1 \theta_1 + \frac{n_1 y_1}{R} = n_2 \theta_2 + \frac{n_2 y_1}{R}$

$\Rightarrow \begin{bmatrix} y_2 \\ n_2 \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(n_2 - n_1) & \frac{R}{R} \end{bmatrix} \begin{bmatrix} y_1 \\ n_1 \theta_1 \end{bmatrix}$

$n_2 - n_1$ — called "Refractive Power" of surface

Refraction Matrix:

$R = \begin{bmatrix} 1 & 0 \\ -(n_2 - n_1) & \frac{R}{R} \end{bmatrix}$

$R$ is the same for $R > 0$ or $R < 0$, $n_2 > n_1$ or $n_2 < n_1$.

Now let's see if we recover the expected results for a thin lens.

Consider two refracting surfaces separated by a very small distance $: n_1{:}n_1$.

Let $P_i = \frac{n_{i+1} - n_i}{R_i}$

$R_2 R_1 = \begin{bmatrix} 1 & 0 \\ -(n_2 - n_1) & \frac{R}{R_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(n_2 - n_1) & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ -(n_2 - n_1) & \frac{R_2}{R} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(n_2 - n_1) & \frac{R_2}{R} \end{bmatrix} = R_1 R_2$
Now $P_i = \frac{1}{f_i}$; where $f_i$ is the focal length of lens (i.e.):

Also, $RT$ matrix multiplication is not commutative, so the order matters.

How do we obtain a ray transfer matrix for a system of lenses?

Input reference plane:

Numbering surfaces:

Numbering reference planes:

$K_{2n+2} = \begin{bmatrix} y_{2n+2} \\ \theta_{2n+2} \end{bmatrix}$

Refraction matrices

Translation matrices

To help us: $RT = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ -p & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & T \\ -p & (1-PT) \end{bmatrix}$

and $TR = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & T \\ 1-PT & 1 \end{bmatrix}$

Note: $R$, $T$ and all products thereof have determinant $= 1$. 
What optical properties can we deduce from \( M \), the system matrix \( \begin{bmatrix} A & B \\ c & D \end{bmatrix} \)?

(a) \( D = 0 \) \( \Rightarrow \) All rays entering \( RP_1 \) at the same point exist the output reference plane at the same angle.

(b) \( B = 0 \) \( \Rightarrow \) All rays entering \( RP_1 \) at the same point exist the output reference plane at the same point.

\[ y_2 = A y_1 \]

\[ y = \tan \theta \approx \theta \]

\[ z = y/\theta \]
This called an "afocal" or "telescopic" system.

\[ \text{output plane} \rightarrow y_2 = B'(n', B') \]

and

\[ \text{output plane} \rightarrow y_2 = B(n, B) \]

\[ A = 0 \Rightarrow y_2 = B(n, B) \Rightarrow \text{rays entering the } \]

\[ \text{output plane} \rightarrow \text{spherical magnification} \]

\[ \text{output plane} \rightarrow \text{parallel to each other.} \]
(e) If either \( A \) or \( D = 0 \), then \( B = -1 \).
If either \( B \) or \( C = 0 \), then \( A = \frac{1}{D} \).

Problem 1

What is location and size of image inside the pool?

\[ P \text{ matrix of surface: } \begin{bmatrix} \frac{1}{R} & 0 \\ -\left(\frac{n^2 - n_1^2}{R}\right) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.2 & 1 \end{bmatrix} \]

matrices leading from image to object is:

\[ M = \begin{bmatrix} 1 & \frac{x}{n} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 15 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{x}{0.78} & (15 - \frac{x}{0.78}) \\ -0.2 & -2 \end{bmatrix} \]

image-to-surface \( P \) at surface-to-object

\[ \text{det } M = -2 + \frac{x}{0.78} + 3 - \frac{x}{3.9} = 1 \]

the object is to be focussed onto the image

\[ \Rightarrow B = 0 = 15 - \frac{x}{0.78} \Rightarrow x = 11.7 \text{ cm.} \]

Lateral (transverse) magnification is: \( A = \frac{1}{D} = -0.5 \)

\( \Rightarrow \) Image is inverted and 1.0 cm long.
**Problem 2**

Given: \( R = 2.4 \text{ cm} \)

1. *Final Image*:
   - \( n = 1.6 \)
   - \( 2.8 \text{ cm} \)

2. Calculation of Image Position:
   
   \[
   M = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(n-1.6) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(1.6-1) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
   \]

   Simplifying:
   
   \[
   M = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5625 & 6.25 \\ -0.391 & -2.56 \end{bmatrix} = \begin{bmatrix} 0.5625 - 0.391 & 6.25 - 2.56 \\ -0.391 & -2.56 \end{bmatrix}
   \]

   Thus, \( x = 2.44 \text{ cm} \).

3. *Magnification*:
   
   \[
   \frac{1}{D} = \frac{1}{2.56} = -0.39
   \]

   So the image is inverted and \( 0.78 \text{ cm} \) tall.

**Problem 3**

Given: \( R = 1.0 \text{ cm} \)

1. *Parallel Beam Enters Spherical Bead*:
   - \( n = 1.4 \)

   Where is light focussed?
\[
M = \begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -0.2 \\
(1.4 - 1.4) & 1
\end{bmatrix}
\begin{bmatrix}
1 & x^2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-(1.4 - 1.4) & 1
\end{bmatrix}
\]

bead-to-image: right face of bead

between faces: object-to-bead

= \begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0.429 & 1.429 \\
-0.571 & 0.429
\end{bmatrix}

= \begin{bmatrix}
0.429 - 0.571x & 1.429 + 0.429x \\
-0.571 & 0.429
\end{bmatrix}
= \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}

Parallel light focussed \Rightarrow A = 0

\Rightarrow x = 0.75 \text{ cm}

other approach:

\[n_1 \theta_1 = 0 \Rightarrow \begin{bmatrix} y_2 \\ n_2 \theta_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} a y_1 \\ c y_1 \end{bmatrix}\]

Ray \[\begin{bmatrix} y_2 \\ n_2 \theta_2 \end{bmatrix}\] cuts axis at:

\[-\frac{n_2 y_2}{n_2 \theta_2} = -\frac{A}{c}\]

From matrix for bead:

\[
\begin{bmatrix}
0.429 & 1.429 \\
-0.571 & 0.429
\end{bmatrix}
= -\frac{A}{c} = 0.75 \text{ cm}
\]
Slide 2" tall located 10.5 ft, from screen. Find lens focal length that will project an image 40" on the screen, and determine location of lens.

\[ M = \begin{bmatrix} 1 & 10.5 \cdot x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -p & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \]

The real image from a real object will be inverted, so the magnification must be \[ m = -\frac{40}{2} = -20. \]

Then \[ A = \frac{1}{D} = -20 \] and \[ B = D \]

\[ D = 1 - px = -0.05 \]
\[ B = x + 0.05(10.5 - x) = 0 \]

\[ \Rightarrow x = 0.5 \text{ ft}. \]
\[ D = 1 - 0.5p = -0.05 \Rightarrow p = 2.1 \text{ ft}. \]
\[ \Rightarrow f = \text{focal length} = \frac{1}{p} = 5.7 \text{ inches} \]

So a 5.7" focal length lens should be placed 6" from a slide.

Problem 5: \( f_1 = +8 \text{ cm}, \ f_2 = -12 \text{ cm}, \ f_\text{p} = +12.5 \text{ cm} \)

3 cm. \( \uparrow \)

\[ p_1 = 0.25 \text{ cm}, \ p_2 = -8.33 \text{ cm} \]

\[ f_\text{p} = \frac{1}{p_1} = \frac{1}{0.25} = 4 \text{ cm} \]

\[ f_\text{p} = f_1 = +8 \text{ cm} \]

\[ f_\text{p} = f_2 = -12 \text{ cm} \]

\[ p_2 = -8.33 \text{ cm} \]

\[ \Rightarrow \text{ lens} \]

\[ \Rightarrow I \]

\[ \Rightarrow x \]
An object 3 cm tall is located an axis 24 cm to left of +8 cm lens, which is 6 cm to left of -12 cm lens. Find the position and size of the image.

Units: Meters and diopters.

\[ f_1 : \frac{100}{8} = +12.5 \text{ diopters} \]
\[ f_2 : \frac{100}{-12} = -8.33 \text{ diopters} \]

\[ \frac{1}{M} = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 8.33 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.06 & 1 \\ 12.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.24 \\ 0 & 1 \end{bmatrix} \]

\[ M = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.25 & 0.12 \\ -10.42 & -1 \end{bmatrix} \]

\[ = \begin{bmatrix} 0.25 -10.42 X & 0.12 -X \\ -10.42 & -1 \end{bmatrix} \]

To focus object onto image: \( B = 0 \) \( \Rightarrow \) \( X = +0.12 \) meter

Magnification = \( \frac{1}{B} = -1 \)

So object image is 3 cm tall, inverted, located 12 cm to right of 2nd lens.

---

Problem 6

If object displaced axially by \( (dv) \), calculate the displacement \( (dv) \) of the image. Express \( \frac{dv}{dy} \) (longitudinal magnification) in terms of lateral magnification.
Note: (1) If the image is real, \( v \) will be negative.
(2) There are some object distances for which the image is virtual and thus \( v \) is positive.

\[
N = \begin{bmatrix} 1 & -v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/p & 1 \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + pv & u - v + pu \\ -p & 1 - pu \end{bmatrix}
\]

\( \text{lens-to-image} \)

\( \text{(v assumed < 0)} \)

\( B = 0 \) to focus object \( \Rightarrow V = \frac{u}{1 - pu} \)

\( \Rightarrow \frac{dv}{du} = \frac{1}{(1 - pu)^2} \Rightarrow \frac{dv}{du} = M_T^2 \)

\( M_T = \text{Lateral magnification} = \frac{1}{D} = \frac{1}{(1 - pu)} \Rightarrow \frac{dv}{du} = M_T^2 \)

Also: \( \frac{1}{u} - \frac{1}{v} = p = \frac{1}{f} \)

---

**Problem 7:** Show that the distance between real object and real image formed by thin positive lens cannot be \( < 4x \) focal length.

\[
\text{RP}_1 \quad \mathbf{P} \quad \text{RP}_2
\]

For problem 6: \( u - v + pu \cdot v = 0 \) \( \text{recold: } v \text{ is negative} \)

\( K = u - v \)

\( \Rightarrow u - (u - K) + pu(u - K) = 0 \Rightarrow K = \frac{u}{pu - 1} + u \)

\( \Rightarrow \frac{dK}{du} = 1 - \frac{1}{(pu - 1)^2} \)

For \( \frac{dK}{du} + c = 0 \) \( \Rightarrow pu - 1 = 1 \Rightarrow pu = 2 \)

\( pu - 1 \Rightarrow pu = 0 \text{ (trivial)} \)
Consider \( Pu = 2 \) case:
\[
\frac{d^2 K}{d u^2} = \frac{2P}{(Pu-1)^2} = 0 \Rightarrow K \text{ has a minimum}
\]

Let \( u = \frac{2}{P} \Rightarrow K_{min} = \frac{2P}{1} + \left( \frac{2}{P} \right) = \frac{4}{P} = 4f
\]

(recall: \( f = \frac{1}{P} \))

### 5.52

**Locating cardinal points**

Suppose you know \( M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) that links an input plane \( RP_1 \) to an output plane \( RP_2 \). We want to know the location of the two focal points, the principal planes of unit transverse magnification and the nodal planes of unit angular magnification. Let \( n_1, n_2 \) represent the refractive index to the left and to the right of the system.

For the above ray: \( n_1 \theta_1 = 0 \)

\( \Rightarrow \) at \( RP_2 \): the ray transfer matrix \( \Rightarrow y_2 = Ay_1 \)

and \( \theta_2 = \frac{C y_1}{n_2} \)
Distance from $RP_2$ to $F_2$ is $-\frac{y_2}{\theta_2} = -\frac{n_2 \cdot A}{c}$

which locates the 2nd focal point.

Distance from $RP_2$ to $H_2$ (2nd principal plane) = $\frac{n_2(1-A)}{c}$

Now consider:

**1st focal point**

Starting from $F_1$, the ray is parallel to the axis from $H_1$ onward $\Rightarrow \theta_2 = n_2 \theta_2 = 0$

$\Rightarrow n_2 \theta_2 = cy_1 + Dn_1 \theta_1 = 0$

$\Rightarrow y_1 = -Dn_1 \theta_1$

$\Rightarrow y_1 = -\frac{Dn_1}{c}$

which locates

Distance from $F_1$ to $RP_1 = -\frac{y_1}{\theta_1} = \frac{n_1 \cdot D}{c}$

Distance from $F_1$ to $RP_1$

the 1st focus.

At $H_1$: y-coordinate $= y_2 = Ay_1 + Bn_1 s_1$

$\Rightarrow f_1 = y_2 = n_1 (DA/c) + Bn_1 = -\frac{n_1 (AD - BC)}{c}$

Distance from $RP_1$ to $H_1 = \frac{n_1 (D-1)}{c}$
what of the nodal points? Recall: Nodal points L₁ and L₂ have the property that any ray entering system directed toward L₁ appears after system to be a ray coming from L₂, making the same angle \( \theta_1 = \theta_2 \). Pictorially:

\[
\begin{align*}
    \text{RP}_1 & \quad \text{L₁} \quad \text{L₂} \\
    \text{RP}_2 & \quad \text{P₁} \quad \text{P₂}
\end{align*}
\]

What is the matrix chain linking L₂ back to L₁?

\[
\begin{bmatrix}
    1 & \frac{P_2}{n_2} \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    A & B \\
    C & D
\end{bmatrix}
\begin{bmatrix}
    1 & \frac{-P_1}{n_1} \\
    0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
    A + \left( \frac{P_2 C}{n_2} \right) & -(\frac{AP_1}{n_1}) + B + \frac{P_2}{n_2} \left( \frac{-CP_1}{n_1} \right) + D \\
    C & \frac{-CP_1}{n_1}
\end{bmatrix}
\]

call this matrix \( \phi \). Let \((y_0, \theta_0)\) = coordinates of a ray entering the L₁ plane and \((y_3, \theta_3)\) the coordinates of a ray leaving the L₂ plane.

\[
\begin{align*}
    y_3 &= \phi_{11} y_0 + \phi_{12} n_1 \theta_0 \\
    n_2 \theta_3 &= \phi_{21} y_0 + \phi_{22} n_1 \theta_0
\end{align*}
\]

L₁ and L₂ are nodal points => \( y_0 = 0 \)

and \( y_3 = 0 \) for all \( \theta_0 \) values,

and \( \theta_3 = \theta_0 \).
\[ \Rightarrow \phi_{12} = 0 \]

and

\[ \phi_{22} \left( \frac{n_1}{n_2} \right) = 1 \Rightarrow P_1 = \frac{(Dn_1 - n_2)}{c} \]

\[ \Rightarrow P_2 = \frac{(n_1 - An_2)}{c} \]

<table>
<thead>
<tr>
<th>System Parameter</th>
<th>From</th>
<th>To</th>
<th>Function of Matrix Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Focal point</td>
<td>RP_1</td>
<td>F_1</td>
<td>n, D/c</td>
</tr>
<tr>
<td>1st Focal length</td>
<td>F_1</td>
<td>H_1</td>
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</tr>
<tr>
<td>1st Principal point</td>
<td>RP_1</td>
<td>H_1</td>
<td>n, (D-1)/c</td>
</tr>
<tr>
<td>1st Nodal point</td>
<td>RP_1</td>
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<td>RP_2</td>
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<td>-n_2/A/c</td>
</tr>
<tr>
<td>2nd Focal length</td>
<td>H_2</td>
<td>F_2</td>
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<tr>
<td>2nd Principal point</td>
<td>RP_2</td>
<td>H_2</td>
<td>n_2 (1-A)/c</td>
</tr>
<tr>
<td>2nd Nodal point</td>
<td>RP_2</td>
<td>L_2</td>
<td>(n_1 - An_2)/c</td>
</tr>
</tbody>
</table>

What happens to \([A \ B] \) matrix if it operates between two principal planes?

New matrix is =

\[
\begin{bmatrix}
1 & 1-D \\
0 & c
\end{bmatrix}
\]

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[H_2 \rightarrow \text{RP}_2, \quad \text{RP}_2 \rightarrow \text{RP}_1, \quad \text{RP}_1 \rightarrow H_1\]

So between the two principal planes there is a refractive matrix that is the same as a thin lens of power

\[P = -c = \frac{1}{f} \]

There is an object-to-image relationship between \( \text{RP}_1 \) and \( H \), and \( H_2 \) with unity transverse magnification.
What happens to a \( \begin{bmatrix} A & B \\ c & D \end{bmatrix} \) matrix operating between the two focal planes?

New matrix:
\[
\begin{bmatrix}
1 - \frac{A}{c} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
A & B \\
c & D
\end{bmatrix}
\begin{bmatrix}
1 & -\frac{D}{c} \\
0 & 1
\end{bmatrix}
\]

\( F_2 \to F_2' \quad R P_2 \to R P_1' \quad R P_1 \to -F_1 \)

\[
\begin{bmatrix}
A - \frac{A}{c} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
B \\
c
\end{bmatrix}
= \begin{bmatrix}
0 \\
\frac{1}{f}
\end{bmatrix}
\]

Consequences:

(1) Ray height in either focal plane depends only on the ray angle in the other focal plane.

(2) If \( Z_1 \) = focal distance of an object point measured to the left from \( F_1 \) and \( Z_2 \) = focal distance of corresponding image point, then:

\[ Z_1 Z_2 = -f^2 \]

Additional problems

Problem 8

Glass hemisphere (radius = \( R \)), refractive index \( n \) plane surface faces left. Show that the 2nd principal point is the intersection of the convex surface with the optical axis. Also show that the 1st principal point is inside the lens at a distance \( (R/n) \) from the plane surface. Finally, show that the focal length of the lens is \( \frac{R}{n-1} \).

Locate the reference planes at the two surfaces of the hemisphere.
Matrix chain:

\[
\begin{bmatrix}
1 & 0 \\
-(1-n) & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & R/n \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

Refraction at curved surface
reduced thickness of lens
Refraction by a plane surface

\[
= \begin{bmatrix}
1 & R/n \\
-(n-1) & 1/n \\
\end{bmatrix}
\]

(i) Distance from \( RP_2 \) to the 2nd principal point:

\[
\left(1 - \frac{A}{c}\right) = 0
\]

(ii) Distance from \( RP_1 \) to the 1st principal point:

\[
\frac{(D-1)}{c} = \frac{R}{n}
\]

(iii) Focal length:

\[
\frac{1}{c} = \frac{R}{(n-1)}
\]

Problem 9

Thin lens of positive focal length \((=10\text{cm})\) is separated by 5 cm from a thin negative lens of focal length \((-10\text{cm})\).

Convert distances to meters. Convert focal lengths to diopeters. System matrix \( M \):

\[
\begin{bmatrix}
1 & 0 \\
0.05 & 1 \\
10 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
-10 & 1 \\
\end{bmatrix}
\]
which corresponds to:

\[
\begin{bmatrix}
\text{negative lens at } RP_2 \\
\text{positive lens at } RP_1
\end{bmatrix}
\begin{bmatrix}
\text{lens separation}
\end{bmatrix}
\begin{bmatrix}
0.5 \\
-5
\end{bmatrix}
\begin{bmatrix}
0.05 \\
1.5
\end{bmatrix}
\]

\[= \frac{1}{c} = \pm 0.2 \text{ meter} = 20 \text{ cm}.
\]

\[= \frac{D}{c} = -0.3 \text{ meter to the right of } RP_1.
\]

\[\text{Focal length is } 20 \text{ cm. } \Rightarrow \text{1st principal plane is } 10 \text{ cm. to the left of the positive lens.}
\]

\[\text{Second focal point } = \frac{-A}{c} = 0.1 \text{ meter} = 10 \text{ cm. to the right of } RP_2.
\]

\[\text{The 2nd principal plane } = \frac{(1-A)}{c} = 10 \text{ cm. to the left of the negative lens.}
\]

This combination of lenses is a simple telephoto lens.

---

Now extend the ray transfer method to reflecting systems.

Basic rule: If a light ray is travelling in the z direction, the refractive index of the medium through which it moves is negative.

\[\begin{bmatrix}
\frac{y}{n} \\
\frac{n}{\theta}
\end{bmatrix}
\]

- to interpret, it is \(\theta\), not \((n\theta)\) that is the geometric angle. Consider:

\[
\begin{align*}
\text{RP}_4 & \quad n = -1 \quad \text{RP}_3 \\
\text{RP}_1 & \quad n = 1 \quad \text{RP}_2
\end{align*}
\]
When the ray is reflected, the ray direction is reversed.
The value of \( \mathbf{n} \) is also reversed.
\( \Rightarrow (\mathbf{n} \cdot \mathbf{\theta}) \) is unchanged.
So saying that across any surface \( [\mathbf{y}_2] = [\mathbf{y}_{1\prime}] \)
cancels the law of reflection.

**T matrices**

Going from a plane \( z = z_1 \) to a plane \( z = z_2 \), the
gap value \( (z_2 - z_1) \) is positive if the ray travels in
the +z direction and negative if the ray travels in
the -z direction. For transfer between planes:
the +z direction. For transfer between planes:
\( M_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), \( M_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \),
\( M_{31} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

**R matrices**

Use this term to include the effect of a reflecting
as well as a refracting surface.

What is the power \( P = \left( \frac{n_2 - n_1}{R} \right) \) of a reflecting
surface? Replace \( n_2 \) by \( (-n) \); \( n \) = refractive
index for medium in which reflector is immersed.
\( \Rightarrow P = \left( -\frac{2n}{R} \right) \).

Radius of curvature of a mirror is positive if the
center of curvature is located to the right of the
surface.
Succession of plane surfaces

Suppose light emerges from a plane-parallel glass plate of thickness \( t \) and index \( n \) \& after \( 2k \) internal reflections.

Original transmission

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\( 2k \)-fold "reflection"

\[
\begin{bmatrix}
1 & 2kt/n \\
0 & 1
\end{bmatrix}
\]

The geometry of the emerging rays will be as if the light was transmitted once through a plate \( (2k+1) \) times as thick.

Problem 12

Light enters a solid glass sphere (radius = \( R \), refractive index = \( n \)) reaches the right-hand surface and some of it is reflected back and emerges on the left. Take reference planes at the left-hand surface. Calculate a ray transfer matrix.

(a) The matrix of (a) to refer to reference planes passing through the center of the sphere. Consider particularly the \( n=2 \) case.

(b) Transform the matrix of (a) to refer to reference planes passing through the center of the sphere.
\[ M = \begin{bmatrix} 1 & 0 \\ \frac{-(n-1)}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2R \ln n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{2n}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{2R}{n} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{n} & \frac{-4R}{n} \\ \frac{n-4}{n} & \frac{n}{n} \end{bmatrix} \]

Final refraction at left

return trip in sphere

reflection at right

trip in sphere

Refraction at left

How do we convert \( M \) to refer to reference planes located a distance \( (t) \) to the left of the left-hand surface of the sphere? The effect is to lengthen the optical path between the reference planes. We would represent this by adding the same \( T \)-matrix \([1 \ t]\) at either end of the matrix chain.

But, we want to move the reference planes to the right, which will shorten the optical path between the planes. To move the reference planes to the center of the sphere, both premultiply and post multiply \( M \) by \([1 \ -R]\),

\[ M_1 = \begin{bmatrix} 1 & -R \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{n-4}{n} & \frac{-4R}{n} \\ \frac{n-4}{n} & \frac{n}{n} \end{bmatrix} \begin{bmatrix} 1 & -R \\ 0 & 0 \end{bmatrix} \]

\[ \Rightarrow \] With respect to the center of the sphere, the system matrix is

\[ M_1 = \begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{n-4}{n} & \frac{-4R}{n} \\ \frac{n-4}{n} & \frac{n}{n} \end{bmatrix} \begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix} \]
\[ M_1 = \begin{bmatrix} -1 & -3 \frac{4}{n} \\ -2(2-n) & 0 \\ \frac{n}{nR} & -1 \end{bmatrix} \]

If \( n = 2 \), \( C = 0 \) \implies 

(i) Lateral magnification \[ \frac{y_2}{y_1} = A = -1 \]

(ii) Angular magnification 
\[ \left( \frac{\theta_2}{\theta_1} \right) = \frac{n_1 D}{n_2} = +1 \]

since \( n_2 = -n_1 \) for the returning ray.

\[ \Rightarrow \]

\[
\begin{array}{c}
\text{retroreflector} \\
\text{RP}_1 \\
\text{RP}_2 \\
n=2
\end{array}
\]
If you choose $R_{P1}$ and $R_{P2}$ at the same place, there is some inherent symmetry.

Show:\[
\begin{bmatrix}
y_2' \\
n_2 \theta_2'
\end{bmatrix} =
\begin{bmatrix} A & B \\ c & D \end{bmatrix}
\begin{bmatrix}
y_1' \\
n_1 \theta_1'
\end{bmatrix}
\]

Take\[
\begin{bmatrix}
y_2' \\
n_2 \theta_2'
\end{bmatrix}
\]
and send it back to form\[
\begin{bmatrix}
y_1' \\
n_1 \theta_1'
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_1' \\
n_1 \theta_1'
\end{bmatrix} =
\begin{bmatrix} A & B \\ c & D \end{bmatrix}
\begin{bmatrix}
y_1' \\
n_1 \theta_1'
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_1' \\
n_1 \theta_1'
\end{bmatrix} =
\begin{bmatrix} A & B \\ c & D \end{bmatrix}
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\begin{bmatrix}
y_2' \\
n_2 \theta_2'
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_2' \\
n_2 \theta_2'
\end{bmatrix} =
\begin{bmatrix} A & B \\ c & D \end{bmatrix}
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\begin{bmatrix} y_1' \\
n_1 \theta_1'
\end{bmatrix}
\]
\[
\begin{align*}
\Rightarrow \begin{bmatrix} y_1^2 \\ n_z \theta_2 \end{bmatrix} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ n_z \theta_2 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} y_1 \\ n_z \theta_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1^2 \\ (n_z \theta_2) \end{bmatrix} \\
\Rightarrow \begin{bmatrix} y_1 \\ n_z \theta_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ n_z \theta_2 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} y_1 \\ n_z \theta_2 \end{bmatrix} &= \begin{bmatrix} A^2 - BC & B(A-D) \\ -C(A-D) & D^2 - BC \end{bmatrix} \begin{bmatrix} y_1 \\ n_z \theta_2 \end{bmatrix}
\end{align*}
\]

This ray must emerge in the \(-z\) direction and along the path of \(n_z \theta_2\).

\[\Rightarrow A = D\]

Let us refer to Hecht.

Ray angles:
\[
\theta_i = \theta_r \Rightarrow \tan(\alpha_i - \theta_i) = \frac{y_i}{R} \sim (\alpha_i - \theta_i)
\]

Take these angles, and \(y_i\) as positive. However, \(R\) is negative \(\Rightarrow a\ mistake\)!

Instead: \((\alpha_i - \theta_i) \equiv -\left(\frac{y_i}{R}\right)\)
\( \alpha_i = \alpha_r + 2\theta_i \)

\( \theta_i = \frac{(\alpha_i - \alpha_r)}{2} \)

\( \Rightarrow \quad n\alpha_r = -n\alpha_i - \frac{2ny_i}{R} \quad (n = \text{refractive index}) \quad \text{of medium} \)

2nd equation:

\( y_r = y_i \)

\[ \begin{bmatrix} y_r \\ n\alpha_r \end{bmatrix} = \begin{bmatrix} \frac{1}{R} & 0 \\ -2n & -1 \end{bmatrix} \begin{bmatrix} y_i \\ n\alpha_i \end{bmatrix} \]

So transfer equation for a mirror is:

\[ \begin{bmatrix} 1 & 0 \\ -\frac{2n}{R} & -1 \end{bmatrix} \]

Special case: Flat mirror \((R \to \infty)\)

\[ M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

Optical cavity:

\[ \begin{array}{c}
\text{System matrix: } M_2 T_{21} M_1 T_{12} \\
\end{array} \]