Consider the following probability distribution functions (PDF)

a) Laplace distribution
\[ g(x) = \frac{1}{2a} \exp\left(-\frac{|x|}{a}\right) \] (3 points)

b) Rayleigh distribution
\[ g(x) = \frac{x}{a^2} \exp\left(-\frac{x^2}{2a^2}\right), \quad x \geq 0 \] (3 points)

For each of the PDFs find mean, variance and the characteristic function.

Hint: Mean = \( \langle x \rangle \) - first moment

Variance = \( \langle x^2 \rangle - \langle x \rangle^2 \), where

\( \langle x^2 \rangle \) is the second moment

Characteristic function is the generator of moments of the distribution. It is simply a Fourier transform of the PDF.
Moments of the PDF are expectation values for powers of the random variable:

\[ M_n = \langle x^n \rangle = \int dx \, p(x) \, x^n \]

However, sometimes it is easier to find moments of PDF by expanding characteristic function in powers of \( \kappa \):

\[ f(\kappa) = \sum_{n=0}^{\infty} \frac{(-i\kappa)^n}{n!} \langle x^n \rangle \]

(2) Gaussian integrals (6 points)

a) Show that \( I = \int_{-\infty}^{\infty} dx \, e^{-x^2/2a} = \sqrt{2\pi a} \).

b) Show also that

\[ \langle x^2 \rangle = a \]
\[ \langle x^4 \rangle = 3a^2 \]
\[ \langle x^6 \rangle = 15a^3 \]

for normalized Gaussian distribution.
3) Show that fluctuations are suppressed by the order $1/N$, where $N$ is the number of particles (3 points).

4) Consider the velocity of a gas particles in one dimension ($-\infty < v < +\infty$).

Find the probability distribution $p(v)$ subject only to the constraint that the average speed is $c$, that is $\langle |v| \rangle = c$. (3 points)

Total 18 points.