The theory of white dwarf stars

Russel - Hertzprung diagram

Brightness

\[ \text{Red giants} \]

\[ \text{Main sequence} \]

\[ \text{White dwarf} \]

\[ \text{White color} \]

\[ \text{Red color} \]

Empirical rule:

The brightness of a star is proportional to its color. The proportionality constant is roughly the same for all stars.

However, there are exceptions:

Red giant stars - huge stars which are too bright for their color.

White dwarfs - small stars, which are too faint for their color.

We consider white dwarfs, because they are good examples of degenerate Fermi gas.

The detailed study shows that white dwarfs are so faint, because the hydrogen supply, which is the most energy source of stars, has been used up, and that now they are composed mainly of helium. The little brightness which they have is due to the gravitational energy release through a slow contraction of the star.

One of the nearest stars to the solar system, the companion of Sirius, 8 light years from us, is a white dwarf.
A bit of History...

Sirius is one of the brightest star in the sky. F.W. Bessel - a famous German astronomer and mathematician - went to obtain precise positional measurements of Sirius. His observations revealed that Sirius was slowly moving as if it were being pulled around in orbit by the gravity of another star. He was convinced that Sirius has an unseen companion. The orbital period of the two stars around each other turned out to be 50 years. The companion star was discovered only after Bessel died, but it was very faint, almost 1000 fainter than Sirius.

Around 1915, Walter Adams obtained the spectrum of companion star and was astonished to find that the faint star was nearly 3 times hotter than Sirius. Using the laws of physics, the scientists calculate the size of star.

To be so hot and yet so faint, the companion
of Sirius had to be as small as the Earth and has a mass of the Sun. It had to be three million times more dense than water. This was a big puzzle, because obviously the companion of Sirius was made of some strange new form of matter, far beyond human experience. The nature of such dense objects, now called white dwarfs, remained a complete mystery until the development of quantum mechanics.

Ok, let us unveil the mystery.
White dwarf content:
mostly helium
Density $\approx 10^7 \text{g/cm}^3$
Mass $\approx 10^{33} \text{g}$
Central temperature $\approx 10^7 \text{K}$

Hence the helium atoms are expected to be completely ionized; the star may be regarded as gas composed by helium nuclei and electrons.

Let's consider this electron gas of electrons as an ideal Fermi gas with a density $\approx 10^{30} \text{electrons/cm}^3$.

$$E_F = \frac{\hbar^2}{2m} \frac{1}{v^{3/2}} \approx 20 \text{ MeV}$$

$$T_F \approx 10^{11} \text{ K}$$

Since the Fermi temperature $T_F > T_{\text{star}}$, the electron gas is a highly degenerate Fermi gas, which behaves as an ideal gas at $T=0$.

The enormous pressure exerted by electron gas is compensated by the gravitational attraction that binds the star. Note that gravitational force is due almost entirely to the helium nuclei.
Idealized Model

White dwarf = \( N \) electrons in their ground state at such density that electrons must be treated by relativistic dynamics. Electrons move in a background of \( N/2 \) motionless Helium nuclei.

- Let us calculate the pressure exerted by a Fermi gas of relativistic particles

\[
E_{p_{\text{fs}}} = \sqrt{(pc)^2 + (mc^2)^2} \quad \text{single-particle energy}
\]

The ground state energy is

\[
E_0 = 2 \sum_{|p| < p_F} \sqrt{(p_0^2 + (mc^2)^2)} = \frac{2 \sqrt{2} \pi}{(2\pi\hbar)^3} \int_0^{p_F} dp 4\pi p^2 \sqrt{(pc)^2 + (mc^2)^2}, \quad \text{where}
\]

\[
p_F = \frac{\hbar}{m} \left( \frac{3\pi^2}{2} \right)^{1/3} \left( \frac{2 \sqrt{2} \pi}{(2\pi\hbar)^3} \frac{4}{3} \pi p_F^3 \right) = N
\]

Changing variables of integration: \( \chi = \frac{p}{mc} \)
\[ E_0 = \frac{m_e^4 c^5}{\pi^2 \hbar^3} \nu \bar{f}(\xi_F), \text{ where} \]

\[ \bar{f}(\xi_F) = \int_0^{\xi_F} dx \frac{x^2}{1+x^2} = \begin{cases} 
\frac{1}{3} \xi_F^3 \left(1 + \frac{3}{10} \xi_F^2 + \cdots\right), & \text{if } \xi_F \ll 1 \\
\frac{1}{4} \xi_F^4 \left(1 + \frac{1}{\xi_F^2} + \cdots\right), & \text{if } \xi_F \gg 1
\end{cases} \]

The total mass and the radius of the star:

\[ M = (m_e + 2m_p)N \sim 2m_p N \iff N = \frac{M}{2m_p} \]

\[ R = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \iff V = \frac{4}{3} \pi R^3 \]

\[ \nu = \frac{V}{N} = \frac{8\pi}{3} \frac{m_p R^3}{M} \]

\[ \xi_F = \frac{p_F}{m_e c} = \frac{\hbar}{m_e c} \left(\frac{3\pi}{2}\right)^{\frac{1}{3}} = \frac{\bar{M}^{\frac{1}{3}}}{\bar{R}}, \text{ where we define} \]

\[ \bar{M} = \frac{9\pi}{8} \frac{M}{m_p}, \quad \bar{R} = \frac{R}{\hbar/m_e c} \]

Now we can obtain pressure:
\[ P_0 = -\frac{\partial E_0}{\partial \nu} = \frac{m e^2 c^5}{\pi^2 a_0^3} \left( -f(x_F) - \frac{\partial f(x_F)}{\partial x_F} v \frac{\partial x_F}{\partial \nu} \right) \]

\[ = \frac{m e^2 c^5}{\pi^2 a_0^3} \left( \frac{1}{3} x_F^3 \sqrt{1 + x_F^2} - f(x_F) \right) \]

The nonrelativistic and extreme relativistic limits of \( P_0 \) are given by

\[ P_0^{\text{nonrel}} \approx \frac{m e^2 c^5}{15 \pi^2 a_0^3} x_F^5 = \frac{4}{5} \frac{\overline{M}^{5/3}}{R^5} \quad (x_F \ll 1) \]

\[ P_0^{\text{rel}} \approx \frac{m e^2 c^5}{12 \pi^2 a_0^3} (x_F^4 - x_F^2) \approx K \left( \frac{\overline{M}^{4/3}}{R^4} - \frac{\overline{M}^{2/3}}{R^2} \right) \quad (x_F \gg 1) \]

where \( K = \frac{m e^2 c^2}{12 \pi^2} \left( \frac{m e c}{\hbar} \right)^3 \)

\[ P_0 \quad -\text{nonrelativistic} \]

\[ R \]
The condition of equilibrium of the star can be obtained as follows.

1) No gravitation. The amount of work that an external force has to do to compress the star of a given mass to a state with defined density is

\[ W = - \int_{0}^{R} P_0 4\pi r^2 dr \]

2) In the presence of gravitation:

\[ E_{gr} = -\frac{\gamma M^2}{R} \] where \( \gamma \) is grav. constant and \( \alpha = \text{const} \neq 1 \).

\[ = \int_{0}^{R} P_0 4\pi r^2 dr = -\frac{\gamma M^2}{R} \]

Differentiating (2) with respect to \( R \), we obtain the condition for equilibrium

\[ P_0 = \frac{2}{\gamma \pi} \frac{\alpha M^2}{R^4} = \frac{\gamma}{4\pi} \sigma \left( \frac{8\pi \rho}{9\pi} \right)^2 \left( \frac{m c^2}{h} \right)^4 \frac{M^2}{R^4} \]

We now determine the relation between \( M \) and \( R \) by inserting an appropriate expression for \( P_0 \).
a) Suppose the temperature of the gas, $T \gg T_F$. Then the electron gas may be considered as Boltzmann gas.

$$P_0 = \frac{N T}{V} = \frac{T}{\nu} = \frac{3 T}{8 \pi m_p R^3}$$

Substitute this into (1).

This case, however, is never applicable for white dwarf star.

b) Suppose the electron gas is a low density. Then it can be considered as nonrelativistic ($x_1 = c < 1$).

Then, $P_0 = P_0^{\text{nonrel}}$ and the equilibrium condition (1) gives

$$\frac{y}{5} \frac{K^*}{R^5} \equiv K' \frac{M^2}{R^4}$$,

where

$$K' = \frac{\alpha}{4 \Pi} \gamma \left( \frac{8 m_p}{9 \pi} \right)^2 \left( \frac{m e c}{\hbar} \right)^4$$

$$= \frac{M^{1/3}}{R} = \frac{y}{5} \frac{K}{K'}$$, or $M^{1/3} R = \text{const} \frac{K}{K'}$. 

The radius of the star decreases as the mass of the star increases. This condition is valid when the density is low. Hence it is valid for small $M$ and large $R$.

c) Suppose the electron gas is at such a high density that relativistic effects are important ($\kappa \gg 1$).

Then, $p_0 = p_0^{\text{rel}}$. The equilibrium condition (1) becomes

$$\kappa \left( \frac{\bar{M}^{2/3}}{R^4} - \frac{\bar{M}^{2/3}}{R^2} \right) = \kappa' \frac{\bar{M}^2}{R^4},$$

or

$$\bar{R} = \bar{M}^{2/3} \sqrt{1 - \left( \frac{\bar{M}}{\bar{M}_0} \right)^{2/3}},$$

where

$$\bar{M}_0 = \left( \frac{k}{k'} \right)^{3/2}$$

Numerically, if we take $\frac{4\pi}{3} \times 10^{39} \, \text{cm}^3$, we obtain

$$M_0 = \frac{8}{3\pi} \, m_p \, M_0 \approx 10^{33} \, \text{g} \approx M_{\odot}.$$

This formula is valid for high densities or $R \gg 0$. 
Hence our formula is valid near $M_0$.

Our model yields a remarkable prediction that no dwarf star can have mass larger than $M_0$, because otherwise it will give an imaginary radius. The physical reason underlying this result is that if the mass is greater than a certain amount, the pressure coming from the Pauli exclusion principle is not sufficient to support the gas against the gravitational collapse.