4.3 The Method of Undetermined Coefficients

A. The Method.

We now consider the (nonhomogeneous) differential equation

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = F(x), \tag{4.23}$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants but where the nonhomogeneous term F is (in general) a nonconstant function of x. Recall that the general solution of (4.23) may be written

$$y = y_c + y_p,$$

where y_c is the *complementary function*, that is, the general solution of the corresponding homogeneous equation (Equation (4.23) with F replaced by 0), and y_p is a *particular integral*, that is, any solution of (4.23) containing no arbitrary constants. In Section 4.2 we learned how to find the complementary function; now we consider methods of determining a particular integral.

We consider first the method of *undetermined coefficients*. Mathematically speaking, the class of functions F to which this method applies is actually quite restricted; but this mathematically narrow class includes functions of frequent occurrence and considerable importance in various physical applications. And this method has one distinct advantage — when it *does apply*, it is relatively simple!

We first introduce certain preliminary definitions

DEFINITION. We shall call a function a *UC function* if it is

either (1) a function defined by one of the following:

(i) x^n, where n is a positive integer or zero.

(ii) e^{ax}, where a is a constant $\neq 0$.

(iii) $\sin(bx + c)$, where b and c are constants, $b \neq 0$.

(iv) $\cos(bx + c)$, where b and c are constants, $b \neq 0$.

or (2) a function defined as a finite product of two or more functions of these four types.

The method of undetermined coefficients applies when the nonhomogeneous function F in the differential equation is a finite linear combination of UC functions. Observe that given a UC function f, each successive derivative of f is either itself a constant multiple of a UC function or else a linear combination of UC functions.

DEFINITION. Consider a UC function f. The set of functions consisting of f itself and all linearly independent UC functions of which the successive derivatives of f are either constant multiples or linear combinations will be called the *UC set* of f.

Example 4.27. The function f defined for all real x by $f(x) = x^3$ is a UC function. Computing derivatives of f, we find

$$f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6 = 6\cdot 1,$$
$$f^{(n)}(x) = 0 \quad \text{for} \quad n > 3.$$

The linearly independent *UC* functions of which the successive derivatives of f are either constant multiples or linear combinations are those given by

$$x^2, \quad x, \quad 1.$$

Thus the *UC set* of x^3 is the set $S = \{x^3, x^2, x, 1\}$.

Example 4.28. The function f defined for all real x by $f(x) = \sin 2x$ is a UC function. Computing derivatives of f, we find

$$f'(x) = 2\cos 2x, \quad f''(x) = -4\sin 2x, \ldots.$$

The only linearly independent UC function of which the successive derivatives of f are constant multiples or linear combinations is that given by $\cos 2x$. Thus the *UC set* of $\sin 2x$ is the set $S = \{\sin 2x, \cos 2x\}$.

Example 4.29. The function f defined for all real x by $f(x) = x^2\sin x$ is the product of the two UC functions defined by x^2 and $\sin x$. Hence f is itself a UC function. Computing derivatives of f, we find

$$\begin{aligned} f'(x) &= 2x\sin x + x^2\cos x, \\ f''(x) &= 2\sin x + 4x\cos x - x^2\sin x, \\ f'''(x) &= 6\cos x - 6x\sin x - x^2\cos x, \\ &\cdots\cdots\cdot \end{aligned}$$

No "new" types of functions will occur from further differentiation. Each derivative of f is a linear combination of certain of the six UC functions given by $x^2\sin x$, $x^2\cos x$, $x\sin x$, $x\cos x$, $\sin x$, and $\cos x$. Thus the set

$$S = \{x^2\sin x, \quad x^2\cos x, \quad x\sin x, \quad x\cos x, \quad \sin x, \quad \cos x\}$$

is the *UC set* of $x^2\sin x$.

We now outline the method of undetermined coefficients for finding a particular integral y_p of

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = F(x),$$

where F is a finite linear combination

$$F = A_1u_1 + A_2u_2 + \cdots + A_mu_m$$

of UC functions $u_1, u_2, \ldots, u_m$, the A_i being known constants. Assuming the complementary function y_c has already been obtained, we proceed as follows:

(1) For *each* of the UC functions

$$u_1, \ldots, u_m$$

of which F is a linear combination, form the corresponding UC set, thus obtaining the respective sets

$$S_1, S_2, \cdots, S_m.$$

(2) Suppose that one of the UC sets so formed, say S_j, is identical with or completely included in another, say S_k. In this case, we omit the (identical or smaller) set S_j from further consideration (retaining the set S_k).

** (3) We now consider in turn each of the UC sets which still remain after step (2). Suppose now that one of these UC sets, say S_l, includes one or more members which are solutions of the corresponding homogeneous differential equation. If this is the case, we multiply *each* member of S_l by the lowest positive integral power of x so that the resulting revised set will contain no members which are solutions of the corresponding homogeneous differential equation. We now replace S_l by this revised set, so obtained. Note that here we consider one UC set at a time and perform the indicated multiplication, if needed, only upon the members of the one UC set under consideration at the moment.

(4) In general there now remains:
  (i) certain of the original UC sets, which were neither omitted in step (2) nor needed revision in step (3), and
  (ii) certain revised sets resulting from the needed revision in step (3).

  Now form a linear combination of *all* of the elements of *all* of the sets of these two categories, with unknown constant coefficients (*undetermined coefficients*).

(5) Determine these unknown coefficients by substituting the linear combination formed in step (4) into the differential equation and demanding that it identically satisfy the differential equation (that is, that it be a particular solution).

We frankly admit that this outline of procedure may seem unnecessarily complicated. Once it is understood, however, it frees one from the need of considering separately all of the special cases which it covers.

B. Examples

A few illustrative examples, with reference to the above outline, should make the procedure clear. Our first example will be a simple one in which the situations of steps (2) and (3) do not occur.

Example 4.30.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x.$$

The corresponding homogeneous equation is

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$$

and the complementary function is

$$y_c = c_1e^{3x} + c_2e^{-x}.$$

The nonhomogeneous term is the linear combination $2e^x - 10\sin x$ of the two UC functions given by e^x and $\sin x$.

(1) Form the UC set for each of these two functions. We find

$$S_1 = \{e^x\},$$
$$S_2 = \{\sin x, \cos x\}.$$

(2) Note that neither of these sets is identical with nor included in the other; hence both are retained.

(3) Furthermore, by examining the complementary function, we see that none of the functions e^x, $\sin x$, $\cos x$ in either of these sets is a solution of the corresponding homogeneous equation. Hence neither set needs to be revised.

(4) Thus the original sets S_1 and S_2 remain intact in this problem, and we form the linear combination

$$Ae^x + B\sin x + C\cos x$$

of the three elements e^x, $\sin x$, $\cos x$ of S_1 and S_2, with the undetermined coefficients A, B, C.

(5) We determine these unknown coefficients by substituting the linear combination formed in step (4) into the differential equation and demanding that it satisfy the differential equation identically. That is, we take

$$y_p = Ae^x + B\sin x + C\cos x$$

as a particular solution. Then

$$y_p' = Ae^x + B\cos x - C\sin x,$$
$$y_p'' = Ae^x - B\sin x - C\cos x.$$

Actually substituting, we find

$$[Ae^x - B\sin x - C\cos x] - 2[Ae^x + B\cos x - C\sin x] - 3[Ae^x + B\sin x + C\cos x] = 2e^x - 10\sin x$$

or

$$-4Ae^x + (-4B + 2C)\sin x + (-4C - 2B)\cos x = 2e^x - 10\sin x.$$

Equating coefficients of like terms, we obtain the equations

$$\begin{cases} -4A = 2 \\ -4B + 2C = -10 \\ -4C - 2B = 0. \end{cases}$$

From these equations, we find that

$$\begin{cases} A = -\frac{1}{2} \\ B = 2 \\ C = -1, \end{cases}$$

and hence we obtain the particular integral

$$y_p = -\frac{1}{2}e^x + 2\sin x - \cos x.$$

Thus the general solution of the differential equation under consideration is

$$y = y_c + y_p = c_1e^{3x} + c_2e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x.$$

Example 4.31.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}.$$

The corresponding homogeneous equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

and the complementary function is

$$y_c = c_1e^x + c_2e^{2x}.$$

The nonhomogeneous term is the linear combination

$$2x^2 + e^x + 2xe^x + 4e^{3x}$$

of the four UC functions given by x^2, e^x, xe^x, and e^{3x}.

(1) Form the UC set for each of these functions. We have

$$S_1 = \{x^2, x, 1\},$$
$$S_2 = \{e^x\},$$
$$S_3 = \{xe^x, e^x\},$$
$$S_4 = \{e^{3x}\}.$$

(2) We note that S_2 is completely included in S_3, so S_2 is omitted from further consideration, leaving the 3 sets

$$S_1 = \{x^2, x, 1\}, \quad S_3 = \{xe^x, e^x\}, \quad S_4 = \{e^{3x}\}.$$

(3) We now observe that $S_3 = \{xe^x, e^x\}$ includes e^x, which is included in the complementary function and so is a solution of the corresponding homogeneous differential equation. Thus we multiply *each* member of S_3 by x to obtain the revised family

$$S_3' = \{x^2e^x, xe^x\},$$

which contains no members which are solutions of the corresponding homogeneous equation.

(4) Thus there remain the original UC sets

$$S_1 = \{x^2, x, 1\}$$

and

$$S_4 = \{e^{3x}\}$$

and the revised set

$$S_3' = \{x^2e^x, xe^x\}.$$

These contain the six elements

$$x^2, \quad x, \quad 1, \quad e^{3x}, \quad x^2e^x, \quad xe^x.$$

We form the linear combination

$$Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x$$

of these six elements.

(5) Thus we take as our particular solution,

$$y_p = Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x.$$

From this, we have:

$$y_p' = 2Ax + B + 3De^{3x} + Ex^2e^x + 2Exe^x + Fxe^x + Fe^x,$$
$$y_p'' = 2A + 9De^{3x} + Ex^2e^x + 4Exe^x + 2Ee^x + Fxe^x + 2Fe^x.$$

We substitute y_p, y_p', y_p'' into the differential equation for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$, respectively, to obtain:

$$\begin{aligned} &2A + 9De^{3x} + Ex^2e^x + (4E + F)xe^x + (2E + 2F)e^x \\ &\quad - 3[2Ax + B + 3De^{3x} + Ex^2e^x + (2E + F)xe^x + Fe^x] \\ &\quad + 2[Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x] \\ &\qquad = 2x^2 + e^x + 2xe^x + 4e^{3x} \end{aligned}$$

(or $2A - 3B + 2C) + (2B - 6A)x + 2Ax^2 + 2De^{3x} + (-2E)xe^x + (2E - F)e^x$

$$= 2x^2 + e^x + 2xe^x + 4e^{3x}.$$

Equating coefficients of like terms, we have:

$$\begin{cases} 2A - 3B + 2C = 0 \\ 2B - 6A = 0 \\ 2A = 2 \\ 2D = 4 \\ -2E = 2 \\ 2E - F = 1. \end{cases}$$

From this $A = 1$, $B = 3$, $C = \frac{7}{2}$, $D = 2$, $E = -1$, $F = -3$, and so the particular integral is

$$y_p = x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x.$$

The general solution is therefore

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x.$$

Example 4.32.

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 3x^2 + 4\sin x - 2\cos x.$$

The corresponding homogeneous equation is $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 0$, and the complementary function is

$$y_c = c_1 + c_2x + c_3\sin x + c_4\cos x.$$

The nonhomogeneous term is the linear combination

$$3x^2 + 4\sin x - 2\cos x$$

of the three UC functions given by

$$x^2, \sin x, \quad \text{and} \quad \cos x.$$

(1) Form the UC set for each of these three functions. These sets are, respectively,

$$S_1 = \{x^2, x, 1\},$$
$$S_2 = \{\sin x, \cos x\},$$
$$S_3 = \{\cos x, \sin x\}.$$

(2) Observe that S_2 and S_3 are identical and so we retain only one of them, leaving the two sets

$$S_1 = \{x^2, x, 1\}, \quad S_2 = \{\sin x, \cos x\}.$$

(3) Now observe that $S_1 = \{x^2, x, 1\}$ includes 1 and x, which, as the complementary function shows, are both solutions of the corresponding homogeneous differential equation. Thus we multiply each member of the set S_1 by x^2 to obtain the revised set

$$S_1' = \{x^4, x^3, x^2\},$$

none of whose members are solutions of the homogeneous differential equation. We observe that multiplication by x instead of x^2 would not be sufficient, since the resulting set would be $\{x^3, x^2, x\}$, which still includes the homogeneous solution x. Turning to the set S_2, observe that both of its members, $\sin x$ and $\cos x$, are also solutions of the homogeneous differential equation. Hence we replace S_2 by the revised set

$$S_2' = \{x\sin x, x\cos x\}.$$

(4) None of the original UC sets remain here. They have been replaced by the revised sets S_1' and S_2' containing the five elements

$$x^4, x^3, x^2, x\sin x, x\cos x.$$

We form a linear combination of these,

$$Ax^4 + Bx^3 + Cx^2 + Dx\sin x + Ex\cos x,$$

with undetermined coefficients A, B, C, D, E.

(5) We now take this as our particular solution

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx\sin x + Ex\cos x.$$

Then

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + Dx\cos x + D\sin x - Ex\sin x + E\cos x,$$
$$y_p'' = 12Ax^2 + 6Bx + 2C - Dx\sin x + 2D\cos x - Ex\cos x - 2E\sin x,$$
$$y_p''' = 24Ax + 6B - Dx\cos x - 3D\sin x + Ex\sin x - 3E\cos x,$$
$$y_p^{iv} = 24A + Dx\sin x - 4D\cos x + Ex\cos x + 4E\sin x.$$

Substituting into the differential equation, we obtain

$$24A + Dx\sin x - 4D\cos x + Ex\cos x + 4E\sin x + 12Ax^2 + 6Bx + 2C - Dx\sin x + 2D\cos x - Ex\cos x - 2E\sin x = 3x^2 + 4\sin x - 2\cos x.$$

Equating coefficients, we find

$$\begin{cases} 24A + 2C = 0 \\ 6B = 0 \\ 12A = 3 \\ -2D = -2 \\ 2E = 4. \end{cases}$$

Hence $A = \frac{1}{4}$, $B = 0$, $C = -3$, $D = 1$, $E = 2$, and the particular integral is

$$y_p = \frac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x.$$

The general solution is

$$y = y_c + y_p = c_1 + c_2x + c_3\sin x + c_4\cos x + \frac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x.$$

Example 4.33. An Initial-Value Problem. We close this section by applying our results to the solution of the initial-value problem

$$\begin{cases} \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 2e^x - 10\sin x, & (4.24) \\ y(0) = 2 & (4.25) \\ y'(0) = 4. & (4.26) \end{cases}$$

By Theorem 4.1, this problem has a unique solution, defined for all x, $-\infty < x < \infty$; let us proceed to find it. In Example 4.30 we found that the general solution of the differential equation (4.24) is

$$y = c_1e^{3x} + c_2e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x. \tag{4.27}$$

From this, we have

$$\frac{dy}{dx} = 3c_1e^{3x} - c_2e^{-x} - \tfrac{1}{2}e^x + 2\cos x + \sin x. \tag{4.28}$$

Applying the initial conditions (4.25) and (4.26) to Equations (4.27) and (4.28), respectively, we have

$$\begin{cases} 2 = c_1e^0 + c_2e^0 - \dfrac{1}{2}e^0 + 2\sin 0 - \cos 0. \\ 4 = 3c_1e^0 - c_2e^0 - \dfrac{1}{2}e^0 + 2\cos 0 + \sin 0. \end{cases}$$

These equations simplify at once to the following:

$$\begin{cases} c_1 + c_2 = \dfrac{7}{2} \\ 3c_1 - c_2 = \dfrac{5}{2}. \end{cases}$$

From these two equations we obtain

$$\begin{cases} c_1 = \dfrac{3}{2} \\ c_2 = 2. \end{cases}$$

Substituting these values for c_1 and c_2 into Equation (4.27) we obtain the unique solution of the given initial-value problem in the form

$$y = \frac{3}{2}e^{3x} + 2e^{-x} - \frac{1}{2}e^{x} + 2\sin x - \cos x.$$

Exercises

Find the general solution of each of the differential equations in Exercises 1 through 17.

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4x^2.$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 8y = 4e^{2x} - 21e^{-3x}.$

3. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 6\sin 2x + 7\cos 2x.$

4. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 10\sin 4x.$

5. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 4y = \cos 4x.$

6. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 10y = 8xe^{-2x}.$

7. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 5y = 5\sin 2x + 10x^2 - 3x + 7.$

8. $4\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 3y = 3x^3 - 8x.$

9. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 10e^{2x} - 18e^{3x} - 6x - 11.$

10. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4y = 4e^{x} - 18e^{-x}.$