

Appendix A

Measurement Errors and Error Propagation

Physics 321, Electric Circuits and Electronics

1 Accuracy, Uncertainty and Precision

Part of making and reporting a measurement is determining how accurate it might be. Finding that a distance is $10.0000 \text{ cm} \pm 0.0001 \text{ cm}$ can tell you something very different from $10 \text{ cm} \pm 1 \text{ cm}$. In common speech, the words *accuracy* and *precision* are often used interchangeably. However, many scientists like to make a distinction between the meanings of the two words. **Accuracy** refers to the relationship between a measured quantity and the real value of that quantity. The accuracy of a single measurement can be defined as the difference between the measured value and the true value of the quantity. Since in most cases you do not know the true value, you seldom know the true accuracy of your answer. (If you did, you wouldn't be bothering to measure it!) Exceptions to this occur primarily when you are testing an apparatus or new measurement method, and in teaching-labs like 321. Since we often know true values in the 321 laboratory, or we have measured the same quantity in different ways, you have the opportunity to compare the achieved **accuracy** (the difference between the true value and your measured value) or the **consistency** (the difference between your two determinations) with your independently estimated error, as described below.

The words *error* and *uncertainty* are also often used interchangeably. Nevertheless, it is important to be aware of the distinction between the *actual error* in a given measurement, i.e. in the amount by which the measured value differs from the true value, and the **uncertainty** in a measurement, which is an estimate of the **most likely magnitude** of the error. The point is that in experiments we do not always know the true value of the quantity we are measuring, and therefore we cannot determine the *actual* error in a result. However, it is still possible to make an estimate of the **uncertainty** (or the “probable error”) in the measurement based on what we know about the properties of electrical components, measuring instruments etc.

The word **precision** refers to the amount of scatter in a series of measurements of the same quantity, assuming they have been recorded to enough significant figures to show the scatter. You should try to record just enough significant figures to show this scatter in the last figure, or possibly the last two. It is possible for a measurement to be very precise, but at the same time not very accurate! For example, if you measure a voltage using a digital voltmeter that is incorrectly calibrated, the measurements will be precise but inaccurate, i.e., repeated measurements will have the same result to several decimal places, yet all of the measurements will be wrong. Just because a measuring device has a digital reading showing many digits does not guarantee it is accurate!

2 Random and Systematic Uncertainty

Measurement uncertainties can be divided into two distinct classes: **random** (or **statistical**) errors, and **systematic** errors. Systematic errors are things like the voltmeter calibration error mentioned above, or perhaps you made all your length measurements with a metal tape measure that had expanded because you were in a much warmer room than the one in which the tape was

constructed. Systematic errors can be quite difficult to identify and estimate, since you have to understand everything about how your measurement system works.

Perhaps counterintuitively, a random error is usually easier to estimate than a systematic error. Random error is due to a combination of the limited precision to which a quantity can be read from a ruler or meter scale and intrinsic “noise” in the measurement. For example, if a radioactive source that gives an average of one count per second is observed for exactly 100 seconds, you will find that you do not always get exactly 100 counts, even if your count is perfectly accurate (no mistakes). About one-third of the time, you will get fewer than 90 or more than 110 counts, and occasionally you will get fewer than 70 or more than 130 counts (about 0.5% of the time). If you plot the distribution of a large number of 100-second measurements, you will observe a curve called a *Poisson distribution*. Unless the number of counts is very small, this curve will be very close to a *Gaussian distribution* that has a distinctive “bell shape” curve. Most random errors follow this kind of distribution. The expected magnitude of the uncertainty in a measurement is described by the width of the distribution. The range that contains 2/3 of the measurements (± 10 in the counting example above) is called “one-sigma” uncertainty. If the errors follow a bell curve, then 95% of the results will be within $\pm 2\sigma$ and 99.5% within $\pm 3\sigma$. The meaning of σ will become clear below.

You can often estimate random error in a measurement empirically. If you make several independent measurements of some quantity, you can estimate the precision of each individual measurement. The sense of *independent* is important: if you measure a length with a meter stick and on your first try estimate 113.3 mm, you are quite likely to again write down 113.3 mm on subsequent measurements if you use the same approach, even if you can really only estimate to, say, ± 0.2 mm. One way to create a dataset of independent measurements is to have different people make measurements and write them down without looking at each other’s results. Or on your own, you could start from a random point on the ruler each time, even flipping the ruler end-for-end, and then do the necessary subtractions afterwards.

Random (or statistical) error can be formalized. To illustrate, suppose the resistance of a high-precision (0.01%) 1 k Ω resistor is tested by measuring V and I for several different voltage settings using a digital voltmeter and a Simpson VOM. The results are tabulated, as in Table 1. The average value of R for this set of measurements is 1002.4 Ω , so the measurement has an error of 2.4 Ω . The precision of any individual measurement of R can be estimated by calculating the **standard deviation**, σ , which quantifies the width of the distribution of measurements

$$\sigma = \frac{1}{\sqrt{N-1}} \left[\sum_{n=1}^N (x_n - \bar{x})^2 \right] \quad (1)$$

where \bar{x} is the average value of the measurements, x_n , and N is the total number of measurements. For our resistance example, $x_n = R_n$, and the standard deviation is $\sigma = 5.6 \Omega$. Standard deviation σ , can be used as an estimate of the uncertainty (probable error), since any individual measurement has a reasonable probability of being in error by at least that amount. Keep in mind that standard deviation only quantifies random error. The *actual* error in a measurement could be much larger than the standard deviation if there are large systematic errors that affect all the measurements the same way, for example an error in the calibration of the meters used to make the measurements.

Table 1: Data for a Precision 1 k Ω Resistor

n	V (V) [†]	I (A) ^{††}	$V/I = R$ (Ω)		
1	1.000	0.99	1,010	Average, \bar{R}	1,002.4 Ω
2	2.000	1.99	1,005	Standard Deviation, σ	5.6 Ω
3	3.000	3.00	1,000	Actual Error	2.4 Ω
4	4.000	4.02	995	% Error	0.24%
5	5.000	4.99	1,002		

[†] measured with a digital voltmeter; ^{††} measured with a Simpson VOM

3 Propagation of Errors

In many experiments, some quantity, Q , may be determined from a mathematical formula that relates two or more separately measured quantities, $Q = f(x_1, \dots, x_n, \dots, x_N)$, where x_n are the separate quantities, and f is a mathematical function. If each of the x_n varies by an amount δx_n , then to first order, Q will vary as

$$\Delta Q = \sum_{n=1}^N \frac{\partial f}{\partial x_n} \Delta x_n \quad (2)$$

where $\partial f/\partial x_n$ is the partial derivative of f with respect to x_n (all x_i held constant other than $i = n$). Suppose we have estimated the uncertainties in all N measured quantities and would like to estimate the overall uncertainty in Q . We know the expected magnitude of Δx_n , but we anticipate that it is equally likely to be positive or negative, so its average value should be close to zero. We usually try to estimate (or assume we know) the quantity $\sigma_{x_n} = \langle (\Delta x_n)^2 \rangle^{1/2}$, namely the square root of the average of $(\Delta x_n)^2$, i.e. the “root mean square” (rms) value of the expected error. The angle brackets, $\langle \dots \rangle$ mean average value. The process of estimating the overall uncertainty in Q given a formula, f , and the uncertainties in individual measurements is called **error propagation**.

Since $\langle Q \rangle = 0$, we instead estimate its rms

$$\sigma_Q = \langle (\Delta Q)^2 \rangle^{1/2} = \left\langle \left[\sum_{n=1}^N \frac{\partial f}{\partial x_n} \Delta x_n \right]^2 \right\rangle^{1/2} \quad (3)$$

The square of the summation produces terms proportional to $\langle \Delta x_i \Delta x_j \rangle$. The cross terms with $i \neq j$ often vanish, since when Δx_i is positive, Δx_j is equally likely to be positive or negative, which assumes the errors Δx_i and Δx_j are **independent**. The values for terms with $i = j = n$ are $\langle (\Delta x_n)^2 \rangle$, which is $\sigma_{x_n}^2$. If the measurements are *not* independent, cross terms must be included.

There are two cases that, in combination, cover the vast majority of the error propagation problems you will face. Let A and B be two measured (or calculated) quantities, then from Eq. 3

$$\begin{aligned}
\text{Case 1: } Q = A + B \text{ or } Q = A - B, \quad \sigma_Q &= \sqrt{\sigma_A^2 + \sigma_B^2} \\
\text{Case 2: } Q = A \cdot B \text{ or } Q = A/B, \quad \frac{\sigma_Q}{Q} &= \sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}.
\end{aligned} \tag{4}$$

A less common variable combination is $Q = A^n$, in which case $\sigma_Q/Q = n(\sigma_A/A)$. Note that when A and B are subtracted ($Q = A - B$), the errors *add* not subtract! You are encouraged to memorize these formulas, although they are easily derived from Eq. 3. For independent errors, it might be intuitive that separate errors combine as the square root of the sum of squares, or as we say combine “in quadrature”. The largest individual error tends to dominate the overall uncertainty. For mixed cases, like $Q = (A + B)/(C + D)$, first add absolute errors for the numerator and denominator, then convert these to % errors and then add them to get the overall error in Q . Error propagation largely becomes an exercise in converting back and forth from absolute to percentages.

You should be mindful of correlated errors. This happens most often when the same quantity shows up in more than one place, in which case individual errors can be perfectly correlated. Take the trivial case of $Q = 2A$, which can also be written $Q = A + A$. Using the formulae above treating independent errors, there are two different answers for σ_Q depending on which way Q is represented. When starting natively from Eq. 3, there will be a correlated cross term that should be included, and the result will be the same.

A more subtle (and important) example is two resistors in parallel, R_1 and R_2 , with equivalent resistance $R = R_1 R_2 / (R_1 + R_2)$. One can calculate the errors in the numerator and denominator of the equivalent resistance formula separately, but the results cannot be combined assuming the errors are independent, since R_1 and R_2 appear in both the numerator and denominator. Using Eq. 3 directly gives

$$\frac{\sigma_R}{R} = \sqrt{\left(\frac{R}{R_1}\right)^2 \left(\frac{\sigma_{R_1}}{R_1}\right)^2 + \left(\frac{R}{R_2}\right)^2 \left(\frac{\sigma_{R_2}}{R_2}\right)^2} \tag{5}$$

Suppose R_1 has a stated accuracy of $\sigma_{R_1}/R_1 = 0.5\%$ while R_2 is much less accurate with $\sigma_{R_2}/R_2 = 10\%$. It might be intuitive to expect the uncertainty in R_2 will dominate the uncertainty in the parallel combination. As a counter example, a parallel resistor combination is very often used to make a fine adjustment to R_1 by having $R_2 \gg R_1$. One can see that the poor accuracy of R_2 will not matter much if, for the sake of illustration, we take $R_2 = 10R_1$. The second term in Eq. 5 contributes only a small amount of uncertainty since $(R/R_2)^2 \approx (R_1/R_2)^2 = 0.01$. Look for this in the Wheatstone Bridge experiment where you will take advantage of parallel resistors to make a fine adjustment in the “nulling” procedure without adding significant uncertainty.

It is sometimes possible to save effort by carefully looking at the magnitude of numbers or formulas and making approximations. In the case of parallel resistors having $R_2 \gg R_1$, one could safely avoid deriving Eq. 5 by noting that the formula $R = R_1 R_2 / (R_1 + R_2) \approx R_1$ when $R_2 \gg R_1$, and the overall uncertainty is therefore determined primarily by the accuracy of R_1 alone.

4 Suggestions for your Lab Notebook

Whenever possible, your measured values of quantities should be compared with given or theoretical values, and the percent error should be recorded. The error should also be compared with your

estimates of uncertainty. If the error is less than 1 or 2 times your estimated σ , no special comment is required other than your result is “in reasonable agreement” with the accepted value. If you have differences that are more than 2σ off (by chance, less than one time in 20), then you are encouraged to look for possible mistakes in your measurements and/or analysis or discuss possible systematic errors that were not included in your estimate.

In some of the experiments you will be asked to make detailed calculations of the uncertainties in your measurements. But usually this is not required, since the calculations are often long and time consuming to do exactly. However, it is always important for experimenters to have an approximate idea of the uncertainties in their results. With suitable approximations, by ignoring variables that make insignificant contributions and using the results for independent errors, you can do most of the error estimation in your head and record the $\pm\sigma$. The usual convention is to analyze the uncertainty and measurements in the same units and then record the absolute error. Generally one significant figure is adequate for error analysis, and in some cases just being sure to round your result to an appropriate number of significant figures is good enough.

Recommended Reading

A thorough discussion of errors and detailed derivations can be found in *Data Reduction and Error Analysis for the Physical Sciences, 3rd Edition* by Bevington and Robinson (QA278 B48 2003).

2020.09.22