Noise Voltage and Power Distributions

Noise Voltage

The voltage V of random noise has a Gaussian probability distribution

$$P(V) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(\frac{-V^2}{2\sigma^2}\right) ,$$

where $P(V)dV$ is the differential probability that the voltage will be within the infinitesimal range V to $V + dV$ and σ is the root mean square (rms) voltage. The probability of measuring *some* voltage must be unity; that is, any probability distribution must be normalized to unity:

$$\int_{-\infty}^{\infty} P(V)dV = 1$$

To confirm the normalization of our noise distribution, we evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(\frac{-V^2}{2\sigma^2}\right) dV$$

$$= 2\int_{0}^{\infty} \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(\frac{-V^2}{2\sigma^2}\right) dV$$

$$= \frac{2^{1/2}}{\pi^{1/2}\sigma}\int_0^\infty \exp\left(\frac{-V^2}{2\sigma^2}\right)dV$$

Using the definite integral

$$\int_0^\infty \exp(-a^2x^2)dx = \frac{\pi^{1/2}}{2a}$$

with $a^2 = (2\sigma^2)^{-1}$ we get the desired result

$$\int_{-\infty}^\infty P(V)dV = \frac{2^{1/2}}{\pi^{1/2}\sigma}\frac{\pi^{1/2}2^{1/2}\sigma}{2} = 1$$

The rms (root mean square) Σ of a normalized distribution is defined by

$$\Sigma^2 = \langle V^2\rangle - \langle V\rangle^2$$

For the symmetric Gaussian distribution, $\langle V\rangle = 0$ so

$$\Sigma^2 = \langle V^2\rangle = \int_{-\infty}^\infty V^2 P(V)dV$$

$$\Sigma^2 = 2\int_0^\infty V^2 \frac{1}{(2\pi)^{1/2}\sigma}\exp\left(\frac{-V^2}{2\sigma^2}\right)dV$$

$$\Sigma^2 = \frac{2^{1/2}}{\pi^{1/2}\sigma}\int_0^\infty V^2\exp\left(\frac{-V^2}{2\sigma^2}\right)dV\ .$$

Using the definite integral

$$\int_0^\infty x^2 \exp(-ax^2)dx = \frac{1}{4a}\left(\frac{\pi}{a}\right)^{1/2}$$

with $a = (2\sigma^2)^{-1}$ yields

$$\Sigma^2 = \frac{2^{1/2}}{\pi^{1/2}\sigma}\frac{2\sigma^2}{4}2^{1/2}\sigma\pi^{1/2} = \sigma^2 \ ,$$

demonstrating that σ is really the rms.

Noise Power

A square-law detector multiplies the input voltage V by itself to yield an output voltage $V_o = V^2$ that is proportional to the input power. What is the probability distribution $P_o(V_o)$ of detector output voltage when the input voltage distribution is Gaussian? For simplicity, we set $\sigma = 1$. The same value of V_o is produced by both positive and negative values of V and the probability distribution of V is symmetric, so

$$P_o(V_o)dV_o = 2P(V)dV$$

for $V \geq 0$. Since $dV_o = 2VdV$, $dV/dV_o = V_o^{-1/2}/2$ and

$$P_o(V_o) = \frac{1}{(2\pi)^{1/2}} V_o^{-1/2} \exp(-V_o/2)$$

for $0 \leq V_o < \infty$. Notice that the distribution of detector output voltages is sharply peaked near $V_o = 0$ and has a long exponentially decaying tail, so it looks quite different from a Gaussian distribution.

To confirm that P_o is properly normalized we evaluate

$$\int_0^\infty P_\text{o}(V_\text{o})dV_\text{o} = \frac{1}{(2\pi)^{1/2}} \int_0^\infty V_\text{o}^{-1/2} \exp(-V_\text{o}/2)dV_\text{o}$$

using the definite integral

$$\int_0^\infty x^n \exp(-ax)dx = \frac{\Gamma(n+1)}{a^{n+1}} \ ,$$

where Γ is the Gamma function, $\Gamma(1/2) = \pi^{1/2}$, and $\Gamma(n+1) = n\Gamma(n)$. Subsituting $n = -1/2$ and $a = 1/2$ yields the correct result

$$\int_0^\infty P(V_\text{o})dV_\text{o} = \frac{1}{(2\pi)^{1/2}} \frac{\Gamma(1/2)}{(1/2)^{1/2}} = \frac{1}{(2\pi)^{1/2}} \pi^{1/2} 2^{1/2} = 1 \ .$$

The mean detector output voltage is

$$\langle V_\text{o} \rangle = \int_0^\infty V_\text{o} P_\text{o}(V_\text{o})dV_\text{o}$$

$$= \int_0^\infty \frac{V_\text{o}}{(2\pi)^{1/2}} V_\text{o}^{-1/2} \exp(-V_\text{o}/2)dV_\text{o}$$

$$= \frac{1}{(2\pi)^{1/2}} \int_0^\infty V_\text{o}^{1/2} \exp(-V_\text{o}/2)dV_\text{o} \ .$$

Using the definite integral above with $n = 1/2$ and $a = 1/2$ yields

$$\langle V_\text{o} \rangle = \frac{1}{(2\pi)^{1/2}} \frac{\Gamma(3/2)}{(1/2)^{3/2}} \ .$$

$\Gamma(3/2) = (1/2)\Gamma(1/2) = \pi^{1/2}/2$ so

$$\langle V_{\rm o} \rangle = \frac{1}{(2\pi)^{1/2}} \frac{\pi^{1/2}}{2} 2^{3/2} = 1 \ .$$

The average detector output voltage is nonzero; it equals the average input power. Had we allowed $\sigma \neq 1$ we would have gotten $\langle V_{\rm o} \rangle = \sigma^2$.

What is the rms Σ_0 of the detector output voltage?

$$\Sigma_0^2 = \langle V_o^2 \rangle - \langle V_o \rangle^2$$

so we must evaluate

$$\langle V_o^2 \rangle = \int_0^\infty V_o^2 P_o(V_o) dV_o$$

$$= \frac{1}{(2\pi)^{1/2}} \int_0^\infty V_o^2 V_o^{-1/2} \exp(-V_o/2) dV_o$$

$$= \frac{1}{(2\pi)^{1/2}} \int_0^\infty V_o^{3/2} \exp(-V_o/2) dV_o \ .$$

Using the definite integral above with $n = 3/2$ and $a = 1/2$ yields

$$\langle V_o^2 \rangle = \frac{1}{(2\pi)^{1/2}} \frac{\Gamma(5/2)}{(1/2)^{5/2}} \ .$$

$\Gamma(5/2) = (3/2)\Gamma(3/2) = 3\pi^{1/2}/4$ so

$$\langle V_o^2 \rangle = \frac{1}{(2\pi)^{1/2}} 2^{5/2} \frac{3\pi^{1/2}}{4} = 3$$

and

$$\Sigma_o^2 = \langle V_o^2 \rangle - \langle V_o \rangle^2 = 3 - 1 = 2 \ .$$

Thus the rms $\Sigma_o = 2^{1/2}$ of the detector output voltage is $2^{1/2}$ times the mean output voltage. [If we had kept track of $\sigma \neq 1$, we would have gotten $\Sigma_o = 2^{1/2}\sigma^2$.] The rms uncertainty in each independent sample of the measured noise power is $2^{1/2}$ times the mean noise power. If $N \gg 1$ independent samples are averaged, the fractional rms uncertainty of the averaged power is $(2/N)^{1/2}$. This result is the heart of the radiometer equation. According to the central limit theorem, the distribution of these averages approaches a Gaussian as N becomes large.