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Demonstration of Chaos

Advanced Laboratory, Physics 407
University of Wisconsin
Madison, Wisconsin 53706

Abstract

A simple resonant inductor-resistor-diode series circuit can be used to demonstrate universal properties of bifurcation and chaos. Quantitatively, the measurement of circuit voltages can be used to determine two universal chaos parameters.

1. Introduction

Consider the following recurrence relation:

$$V_{j+1} = V_j(1 - V_j)V_{in}$$

Suppose we chose some starting value for the variable $V_j = V_1$, then use the formula above to generate V_2, then V_3, and so on. For some values of V_{in}, the value of V_j converges to some value V_∞ when j continues toward infinity. And even more surprisingly, V_j does not converge to any value for other V_{in}.[1]

In those nonconvergent cases, as j increases from one large integer to the next, V_j alternates from one apparent value of V_∞ to another. It is as if V_j has two limits as j tends to infinity. For higher values of V_{in}, there may be four, eight, sixteen, or more apparent limits to V_j. These "limits" are called "attractors."

Fig. 1 shows a graph of the attractors versus V_{in}.[2] Notice that at certain values of V_{in}, lines break from one into two parts, each of which breaks into two more parts further to the right. Each break is called a "bifurcation."

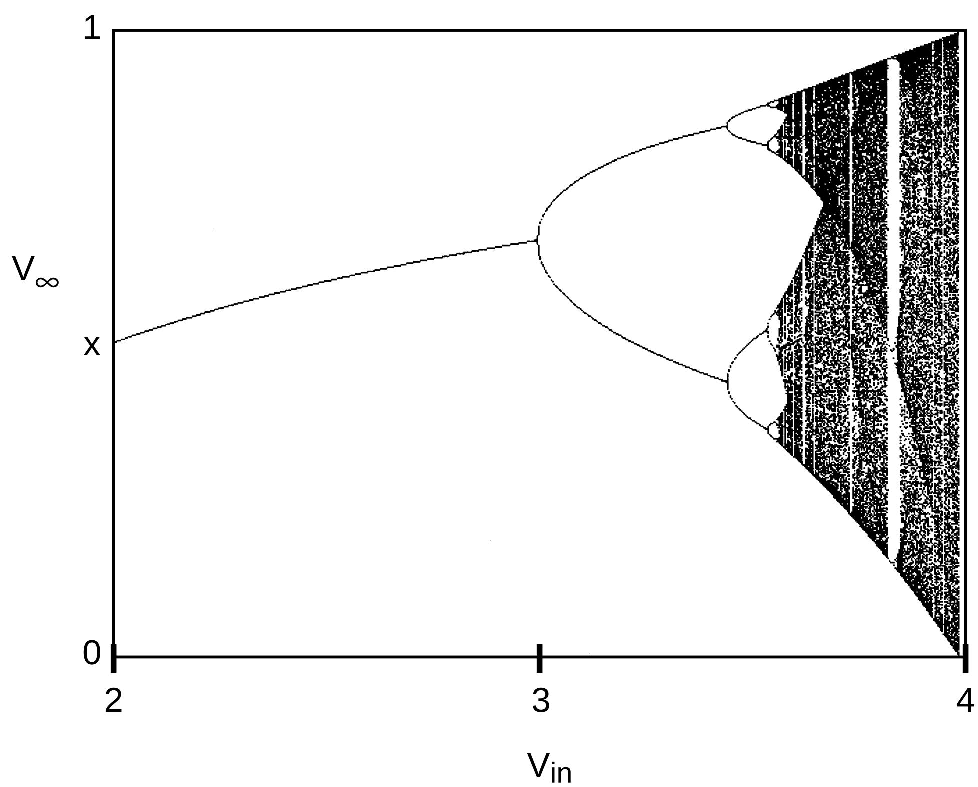

Fig. 1

So many bifurcations occur that individual lines blend into one another. This is called "chaos." Within the chaos region there exist a few places where the graph suddenly displays only a few lines; these "nodes" appear as white against the chaos.

At the first bifurcation, one line splits into two; the horizontal position of this bifurcation will be called x_1, and the vertical position will be called $y_{(1,1)}$ for reasons to follow. At the second bifurcation, the two lines split into four; there are two bifurcation points, one right above another. Call their horizontal position x_2; call the lower point's vertical position $y_{(2,1)}$ and the higher point's vertical position $y_{(2,2)}$. Continuing on, the horizontal position of the third bifurcation will be called x_3, the vertical positions of the bifurcation points will be labelled $y_{(3,1)}$, $y_{(3,2)}$, $y_{(3,3)}$, and $y_{(3,4)}$, from lowest to highest.

Define

$$X_j \equiv \frac{x_j - x_{j-1}}{x_{j+1} - x_j}$$

For example, $X_1 = \frac{x_2 - x_1}{x_3 - x_2}$.

Define

$$Y_j \equiv \frac{\Delta y_j}{\Delta y_{j+1}} \ ,$$

where Δy_j is the largest y splitting of the jth bifurcation and Δy_{j+1} is the largest y splitting of the next set of bifurcations. For example, $Y_3 = \frac{y_{(3,2)} - y_{(3,1)}}{y_{(4,4)} - y_{(4,3)}}$. In this case $y_{(4,4)} - y_{(4,3)}$ is the largest of the splittings. Notice that Y_1 is not defined, since this is the first bifurcation and has no corresponding splitting.

The reason for making these odd definitions is because there exist two constants (called "Feigenbaum parameters") defined in terms of X_j and Y_j as j increases to infinity.

$$\delta \equiv \lim_{j \to \infty} X_j = 4.669...$$

$$\alpha \equiv \lim_{j \to \infty} Y_j = 2.503...$$

It happens that these two values are the same for many recurrence relations similar to the one above.[1] In that sense, they are "universal."

2. Physics of Bifurcations and Chaos

The equation

$$V_{j+1} = V_j(1 - V_j)V_{in}$$

has a physical analog.[2] Consider the circuit shown in Fig. 2.

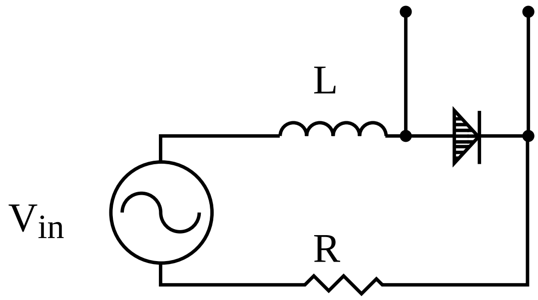

Fig. 2

The V_{in} term in the recurrence relation represents the magnitude of the AC input voltage of frequency f. The V_j term represents the diode voltage when $j \to \infty$. So the voltage across the diode is represented by one of the lines displayed in figure 1.

To be mathematically correct, both V_{in} and V_∞ must be dimensionless quantities. Therefore, V_{in} and V_∞ must equal the input and diode voltages respectively, each divided by some reference voltage. The magnitude of this reference voltage is irrelevant to demonstrating bifurcation and chaos, as it can be shown that the reference voltage must drop out of the formulas for α and δ.

Suppose that we set V_{in} between 1 and x_1. Then according to figure 1, the diode voltage is well-defined and must lie on the line in figure 1. If V_{in} is set between x_2 and x_1, somewhat to the right of the first bifurcation, then the diode voltage will switch rapidly between two values, as predicted in figure 1. Similarly, the diode voltage switches among four, eight, sixteen, thirty-two, etc, values for higher V_{in} until chaos is reached.

The diode voltage can easily be displayed on an oscilloscope triggered from the driving voltage. The oscilloscope trace will show a sinusoidal trace when V_{in} is below the first bifurcation, whereas for higher input voltages, several traces and odd bumps appear on the display. Each of these corresponds to one attractor.

Physically, the input signal is alternately forward and reverse biasing the diode. However the diode has a finite switching speed. Thus, the diode has a "memory time" during which it continues in the previous state (conducting or nonconducting) before switching. This "memory" effect due to the diode capacitance causes the diode voltage to switch from one voltage to another in the bifurcation region on alternate cycles of the input voltage. To see this effect, the frequency of the input volage should be tuned close to the resonant frequency of the LCR series circuit in order to maximize the current through the circuit elements. The C here is the diode capacitance while the L and R are explicit circuit components. Detailed theoretical analyses of the circuit are described in several journal articles.[3][4]

3. Equipment

These are the specifics for the equipment used in the experiment:

- A shielded circuit box contains a 68 ohm resistor, a 10 mH inductor and a 1N4005 Si signal diode. The box has BNC connections for the voltage source and the diode voltage.
- A Tektronix model 2465B (400 MHz) oscilloscope. Detailed documentation about this machine is provided in the packet on top of the oscilloscope.
- A Stanford Research Systems Synthesized Function Generator (model DS345).
- A Wavetek frequency generator (model 184).

Even though the DS345 frequency generator is available, a Wavetek frequency generator still will be useful to explore the properties of the circuit. The Stanford frequency generator requires the user to push buttons to change the voltage and frequency in steps, whereas the Wavetek frequency generator has a knobs controlling the output voltage and the frequency.

4. Experimental Procedure

1. Before the chaos and bifurcations can be explored, f must be set close to the circuit resonant frequency. Since R = 68 Ω, L = 10 mH, and the diode capacitance is C $\approx$ 100 pF the formula f $=1/(2\pi\sqrt{LC})$ gives f $\approx 10^5$ Hz. A frequency of about 60 kHz is recommended. Using the Wavetek function generator, you can scan the frequency slightly to optimize the bifurcation structure. The circuit is shown in Fig. 3.

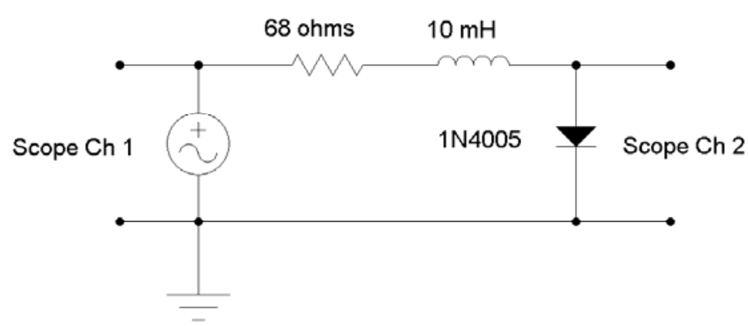

Fig. 3

Slowly increase the input voltage, recording qualitatively what the oscilloscope displays. You should see the bifurcations described in section 2 above. Make sure you have a good feel for what is happening as you vary the input voltage. Vary the frequency slightly to optimize the bifurcations.

2. Set the input voltage somewhat below the first bifurcation voltage x_1, but high enough so that the diode is conducting.

3. You will now gather data so that you may make an approximation of δ and α. Record V_{diode} vs. V_{in} in appropriate steps and slowly increase V_{in} until the first bifurcation is on the verge of occurring. This is x_1. Use the DS345 function generator for data taking and use the scope cursors to make accurate measurements of the input and diode voltages. Voltage amplitudes should be recorded relative to the zero baseline.

4. Using the oscilloscope cursors, measure the vertical distance from the diode baseline to the highest reverse voltage of the diode. This is $y_{(1,1)}$.

5. Estimate how accurately you measure x_1 and $y_{(1,1)}$. Make certain that you can explain why these errors are appropriate.

6. Continue to record V_{diode} vs. V_{in} until the second bifurcation is about to occur. Step sizes of $\Delta v_{in} = 5$ mV are probably sufficient although you should be able to step as small as $\Delta v_{in} = 1$ mV if necessary.

7. Repeat step 6 for higher bifurcations. You may get good data up to the fourth bifurcation. Take representative scope trace pictures in the various bifurcation regions as you go along.

8. You now have values for x and y for all observable bifurcations. These bifurcations, of course, occur at j much lower than infinity; however, the limits for α and δ converge fast enough that you can use these low-order bifurcations to approximately determine the Feigenbaum parameters. Calculate X_j for each bifurcation available (don't forget to propogate the estimated errors from above). Calculate Y_j using each reasonable combination of points, as described in the Special Note of section 1; that is, find each Y_j multiple times, using the lowest two bifurcation points for each, then the third and fourth, then the fifth and sixth, etc.

9. Calculate a weighted mean of the X_j and Y_j for α and δ (respectively); compare each to the theoretical values.

10. The first measurements of α and δ (X_1 and Y_2) will be probably be especially incorrect. Why? Calculate a mean for each Feigenbaum parameter, this time ignoring the first measurements.

11. Returning to the circuit, increase V_{in} until chaos is reached.

12. Further increase the input voltage until you find a node. How many peaks are there in the diode voltage (per function generator cycle)?

13. Continue increasing the input voltage to search for other nodes. Describe the node(s) that you find.

References

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[3] P.S. Linsay, Phys. Rev. Lett. **47**, 1349 (1981).

[4] R. W. Rollins and E. R. Hunt, Phys. Rev. Lett. **49**, 1295 (1982); R. W. Rollins and E. R. Hunt, Phys. Rev. **A29**, 3327 (1982).