

(a) Show that  $e^x$  and  $e^{2x}$  are linearly independent solutions of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

- (b) What is the complementary function of the given nonhomogeneous equation?  
 (c) Show that  $2x^2 + 6x + 7$  is a particular integral of the given equation.  
 (d) What is the general solution of the given equation?

## 4.2 The Homogeneous Linear Equation With Constant Coefficients

### A. Introduction

In this section we consider the special case of the  $n$ th order homogeneous linear differential equation in which all of the coefficients are real constants. That is, we shall be concerned with the equation

$$(4.11) \quad a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

where  $a_0, a_1, \dots, a_{n-1}, a_n$  are real constants. We shall show that the general solution of this equation can be found explicitly.

In an attempt to find solutions of a differential equation we would naturally inquire whether or not any familiar type of function might possibly have the properties which would enable it to be a solution. The differential equation (4.11) requires a function  $f$  having the property such that if it and its various derivatives are each multiplied by certain constants, the  $a_i$ , and the resulting products,  $a_i \frac{d^{n-i} f}{dx^{n-i}}$ , are then added, the result will equal zero. For this to be the case we would need a function such that its derivatives were constant multiples of itself. Do we know of functions  $f$  having this property that  $\frac{d^k}{dx^k}[f(x)] = cf(x)$  for all  $x$ ? The answer is "Yes," for the exponential function  $f$  such that  $f(x) = e^{mx}$ , where  $m$  is a constant, is such that

$$\frac{d^k}{dx^k}[e^{mx}] = m^k e^{mx}.$$

Thus we shall seek solutions of (4.11) of the form  $y = e^{mx}$ , where the constant  $m$  will be chosen such that  $e^{mx}$  does satisfy the equation. Assuming then that  $y = e^{mx}$  is a solution for certain  $m$ , we have:

$$\frac{dy}{dx} = m e^{mx}$$

$$\frac{d^2y}{dx^2} = m^2 e^{mx}$$

⋮

$$\frac{d^n y}{dx^n} = m^n e^{mx}.$$

Substituting in (4.11), we obtain

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$

or

$$e^{mx}(a_0 m^n + a_1 m^{n-1} \cdots + a_{n-1} m + a_n) = 0.$$

Since  $e^{mx} \neq 0$ , we obtain the polynomial equation in the unknown  $m$ :

$$(4.12) \quad a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0.$$

This equation is called the *auxiliary equation* or the *characteristic equation* of the given differential equation (4.11). If  $y = e^{mx}$  is a solution of (4.11) then we see that the constant  $m$  must satisfy (4.12). Hence, to solve (4.11), we write the auxiliary equation (4.12) and solve it for  $m$ . Observe that (4.12) is formally obtained from (4.11) by merely replacing the  $k$ th derivative in (4.11) by  $m^k$ . ( $k = 0, 1, 2, \dots, n$ .) Three cases arise, according as the roots of (4.12) are real and distinct, real and repeated, or complex.

### B. Case 1. Distinct Real Roots

Suppose the roots of (4.12) are the  $n$  distinct real numbers

$$m_1, m_2, \dots, m_n.$$

Then

$$e^{m_1 x}, e^{m_2 x}, \dots, e^{m_n x}$$

are  $n$  distinct solutions of (4.11). Further, using the Wronskian determinant one may show that these  $n$  solutions are linearly independent. Thus we have the following result.

**THEOREM 4.9.** Consider the  $n$ th-order homogeneous linear differential equation (4.11) with constant coefficients. If the auxiliary equation (4.12) has the  $n$  distinct real roots  $m_1, m_2, \dots, m_n$ , then the general solution of (4.11) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x},$$

where  $c_1, c_2, \dots, c_n$  are arbitrary constants.

*Example 4.18.*  $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$

The auxiliary equation is

$$m^2 - 3m + 2 = 0.$$

Hence

$$(m - 1)(m - 2) = 0, \quad m_1 = 1, m_2 = 2.$$

The roots are real and distinct. Thus  $e^x$  and  $e^{2x}$  are solutions and the general solution may be written

$$y = c_1 e^x + c_2 e^{2x}.$$

We verify that  $e^x$  and  $e^{2x}$  are indeed linearly independent.

Their Wronskian is  $W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0$ .

Thus by Theorem 4.4 we are assured of their linear independence.

*Example 4.19.*  $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$ .

The auxiliary equation is

$$m^3 - 4m^2 + m + 6 = 0.$$

We observe that  $m = -1$  is a root of this equation. By synthetic division we obtain the factorization

$$(m + 1)(m^2 - 5m + 6) = 0$$

or

$$(m + 1)(m - 2)(m - 3) = 0.$$

Thus the roots are the distinct real numbers

$$m_1 = -1, m_2 = 2, m_3 = 3,$$

and the general solution is

$$y = c_1e^{-x} + c_2e^{2x} + c_3e^{3x}.$$

### C. Case II. Repeated Real Roots

We shall begin our study of this case by considering a simple example.

*Example 4.20: Introductory Example.* Consider the differential equation

$$(4.13) \quad \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$$

The auxiliary equation is

$$m^2 - 6m + 9 = 0$$

or

$$(m - 3)^2 = 0.$$

The roots of this equation are

$$m_1 = 3, m_2 = 3.$$

(real but *not* distinct).

Corresponding to the root  $m_1$  we have the solution  $e^{3x}$ , and corresponding to  $m_2$  we have the *same* solution  $e^{3x}$ . The linear combination  $c_1e^{3x} + c_2e^{3x}$  of these "two" solutions is clearly *not* the general solution of the differential equation (4.13), for it is *not* a linear combination of two *linearly independent* solutions. Indeed we may write the combination  $c_1e^{3x} + c_2e^{3x}$  as simply  $c_0e^{3x}$ , where  $c_0 = c_1 + c_2$ ; and clearly  $y = c_0e^{3x}$ , involving *one* arbitrary constant, is not the general solution of the given *second-order* equation.

We must find a linearly independent solution; but how shall we proceed to do so? Since

we already know the one solution  $e^{3x}$ , we may apply Theorem 4.7 and reduce the order. We let

$$y = e^{3xu},$$

where  $u$  is to be determined.

$$\text{Then} \quad \frac{dy}{dx} = e^{3x} \frac{du}{dx} + 3e^{3xu},$$

$$\frac{d^2y}{dx^2} = e^{3x} \frac{d^2u}{dx^2} + 6e^{3x} \frac{du}{dx} + 9e^{3xu}.$$

Substituting into equation (4.13) we have

$$\left( e^{3x} \frac{d^2u}{dx^2} + 6e^{3x} \frac{du}{dx} + 9e^{3xu} \right) - 6 \left( e^{3x} \frac{du}{dx} + 3e^{3xu} \right) + 9e^{3xu} = 0$$

or

$$e^{3x} \frac{d^2u}{dx^2} = 0.$$

Letting  $w = \frac{du}{dx}$ , we have the first-order equation

$$e^{3x} \frac{dw}{dx} = 0$$

or simply

$$\frac{dw}{dx} = 0.$$

The general solution of this first-order equation is simply  $w = c$ , where  $c$  is an arbitrary constant. Choosing the particular solution  $w = 1$  and recalling that  $\frac{du}{dx} = w$ , we find

$$u = x + c_0,$$

where  $c_0$  is an arbitrary constant. By Theorem 4.7 we know that for any choice of the constant  $c_0$ ,  $ue^{3x} = (x + c_0)e^{3x}$  is a solution of the given second order equation (4.13). Further, by Theorem 4.7, we know that this solution and the previously known solution  $e^{3x}$  are linearly independent. Choosing  $c_0 = 0$  we obtain the solution

$$y = xe^{3x},$$

and thus corresponding to the *double* root 3 we find the linearly independent solutions

$$e^{3x} \quad \text{and} \quad xe^{3x}$$

of equation (4.13).

Thus the general solution of Equation (4.13) may be written

$$(4.14) \quad y = c_1 e^{3x} + c_2 x e^{3x}$$

or

$$(4.15) \quad y = (c_1 + c_2 x) e^{3x}.$$

With this example as a guide, let us return to the general  $n$ th-order equation (4.11). If the auxiliary equation (4.12) has the *double* real root  $m$ , we would surely expect that  $e^{mx}$  and  $xe^{mx}$  would be the corresponding linearly independent solutions. This is indeed the case. Specifically, suppose the roots of (4.12) are the double real root  $m$  and the  $(n - 2)$  distinct real roots

$$m_1, m_2, \dots, m_{n-2}.$$

Then linearly independent solutions of (4.11) are

$$e^{mx}, xe^{mx}, e^{m_1x}, e^{m_2x}, \dots, e^{m_{n-2}x},$$

and the general solution may be written

$$y = c_1e^{mx} + c_2xe^{mx} + c_3e^{m_1x} + c_4e^{m_2x} + \dots + c_n e^{m_{n-2}x}$$

or

$$y = (c_1 + c_2x)e^{mx} + c_3e^{m_1x} + c_4e^{m_2x} + \dots + c_n e^{m_{n-2}x}.$$

In like manner, if the auxiliary equation (4.12) has the triple real root  $m$ , corresponding linearly independent solutions are

$$e^{mx}, xe^{mx}, \text{ and } x^2e^{mx}.$$

The corresponding part of the general solution may be written

$$(c_1 + c_2x + c_3x^2)e^{mx}.$$

Proceeding further in like manner, we summarize case II in the following theorem:

**THEOREM 4.10.** (i) Consider the  $n$ th order homogeneous linear differential equation (4.11) with constant coefficients. If the auxiliary equation (4.12) has the real root  $m$  occurring  $k$  times, then the part of the general solution of (4.11) corresponding to this  $k$ -fold repeated root is

$$(c_1 + c_2x + c_3x^2 + \dots + c_kx^{k-1})e^{mx}.$$

(ii) If, further, the remaining roots of the auxiliary equation (4.12) are the distinct real numbers  $m_{k+1}, \dots, m_n$ , then the general solution of (4.11) is

$$y = (c_1 + c_2x + c_3x^2 + \dots + c_kx^{k-1})e^{mx} + c_{k+1}e^{m_{k+1}x} + \dots + c_n e^{m_nx}.$$

(iii) If, however, any of the remaining roots are also repeated, then the parts of the general solution of (4.11) corresponding to each of these other repeated roots are expressions similar to that corresponding to  $m$  in part (i).

We now consider several examples.

*Example 4.21.* Find the general solution of

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0.$$

The auxiliary equation

$$m^3 - 4m^2 - 3m + 18 = 0$$

has the roots 3, 3, -2. The general solution is

$$y = c_1e^{3x} + c_2xe^{3x} + c_3e^{-2x}$$

or

$$y = (c_1 + c_2x)e^{3x} + c_3e^{-2x}.$$

**Example 4.22.** Find the general solution of

$$\frac{d^4y}{dx^4} - 5\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0.$$

The auxiliary equation is

$$m^4 - 5m^3 + 6m^2 + 4m - 8 = 0,$$

with roots 2,2,2,-1. The part of the general solution corresponding to the threefold root 2 is

$$y = (c_1 + c_2x + c_3x^2)e^{2x}$$

and that corresponding to the simple root -1 is simply

$$y = c_4e^{-x}.$$

Thus the general solution is

$$y = (c_1 + c_2x + c_3x^2)e^{2x} + c_4e^{-x}.$$

#### D. Case III. Conjugate Complex Roots

Now suppose that the auxiliary equation has the complex number  $a + bi$  ( $a, b$  real,  $i^2 = -1, b \neq 0$ ) as a nonrepeated root. Then, since the coefficients are real, the conjugate complex number  $a - bi$  is also a nonrepeated root. The corresponding part of the general solution is

$$k_1e^{(a+bi)x} + k_2e^{(a-bi)x},$$

where  $k_1$  and  $k_2$  are arbitrary constants. The solutions defined by  $e^{(a+bi)x}$  and  $e^{(a-bi)x}$  are complex functions of the real variable  $x$ . It is desirable to replace these by two *real* linearly independent solutions. This can be accomplished by using Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,*$$

which holds for all real  $\theta$ .

Using this we have:

$$\begin{aligned} & k_1e^{(a+bi)x} + k_2e^{(a-bi)x} \\ &= k_1e^{ax}e^{bix} + k_2e^{ax}e^{-bix} \\ &= e^{ax}[k_1e^{ibx} + k_2e^{-ibx}] \\ &= e^{ax}[k_1(\cos bx + i\sin bx) + k_2(\cos bx - i\sin bx)] \\ &= e^{ax}[(k_1 + k_2)\cos bx + i(k_1 - k_2)\sin bx] \\ &= e^{ax}[c_1\sin bx + c_2\cos bx], \end{aligned}$$

\*We borrow this basic identity from complex variable theory, as well as the fact that  $e^{ax+bi x} = e^{ax}e^{ibx}$  holds for complex exponents.

where  $c_1 = i(k_1 - k_2)$ ,  $c_2 = k_1 + k_2$  are two new arbitrary constants. Thus the part of the general solution corresponding to the nonrepeated conjugate complex roots  $a + bi$  is

$$e^{ax}[c_1 \sin bx + c_2 \cos bx].$$

Combining this with the results of case II, we have the following theorem covering case III.

**THEOREM 4.11.** (i) Consider the  $n$ th-order homogeneous linear differential equation (4.11) with constant coefficients. If the auxiliary equation (4.12) has the conjugate complex roots  $a + bi$  and  $a - bi$ , neither repeated, then the corresponding part of the general solution of (4.11) may be written

$$y = e^{ax}(c_1 \sin bx + c_2 \cos bx).$$

(ii) If, however,  $a + bi$  and  $a - bi$  are each  $k$ -fold roots of the auxiliary equation (4.12), then the corresponding part of the general solution of (4.11) may be written

$$y = e^{ax}[(c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})\sin bx + (c_{k+1} + c_{k+2}x + c_{k+3}x^2 + \cdots + c_{2k}x^{k-1})\cos bx].$$

We now give several examples.

*Example 4.23.* Find the general solution of

$$\frac{d^2y}{dx^2} + y = 0.$$

We have already used this equation to illustrate the theorems of Section 4.1. Let us now obtain its solution using Theorem 4.11. The auxiliary equation  $m^2 + 1 = 0$  has the roots  $m = \pm i$ . These are the pure imaginary complex numbers  $a \pm bi$ , where  $a = 0$ ,  $b = 1$ . The general solution is thus

$$y = e^{0 \cdot x}(c_1 \sin 1 \cdot x + c_2 \cos 1 \cdot x),$$

which is simply

$$y = c_1 \sin x + c_2 \cos x.$$

*Example 4.24.* Find the general solution of

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0.$$

The auxiliary equation is  $m^2 - 6m + 25 = 0$ . Solving it, we find

$$m = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i.$$

Here the roots are the conjugate complex numbers  $a \pm bi$ , where  $a = 3$ ,  $b = 4$ . The general solution may be written

$$y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x).$$

*Example 4.25.* Find the general solution of

$$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 14\frac{d^2y}{dx^2} - 20\frac{dy}{dx} + 25y = 0.$$

The auxiliary equation is

$$m^4 - 4m^3 + 14m^2 - 20m + 25 = 0.$$

The solution of this equation presents some ingenuity and labor. Since our purpose in this example is not to display our mastery of the solution of algebraic equations but rather to illustrate the above principles of determining the general solution of differential equations, we unblushingly list the roots without further apologies.

They are

$$1 + 2i, 1 - 2i, 1 + 2i, 1 - 2i.$$

Since each pair of conjugate complex roots is double, the general solution is

$$y = e^x[(c_1 + c_2x)\sin 2x + (c_3 + c_4x)\cos 2x]$$

or

$$y = c_1e^x\sin 2x + c_2xe^x\sin 2x + c_3e^x\cos 2x + c_4xe^x\cos 2x.$$

#### *E. An Initial-Value Problem*

We now apply the results concerning the general solution of a homogeneous linear equation with constant coefficients to an initial-value problem involving such an equation.

*Example 4.26.* Solve the initial-value problem

$$(4.16) \quad \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$$

$$(4.17) \quad y(0) = -3$$

$$(4.18) \quad y'(0) = -1.$$

First let us note that by Theorem 4.1 this problem has a unique solution defined for all  $x$ ,  $-\infty < x < \infty$ . We now proceed to find this solution; that is, we seek the particular solution of the differential equation (4.16) which satisfies the two initial conditions (4.17) and (4.18). We have already found the general solution of the differential equation (4.16) in Example 4.24.

It is

$$(4.19) \quad y = e^{3x}(c_1\sin 4x + c_2\cos 4x).$$

From this, we find

$$(4.20) \quad \frac{dy}{dx} = e^{3x}[(3c_1 - 4c_2)\sin 4x + (4c_1 + 3c_2)\cos 4x].$$

We now apply the initial conditions. Applying condition (4.17),  $y(0) = -3$ , to Equation (4.19), we find

$$-3 = e^0(c_1\sin 0 + c_2\cos 0)$$



which reduces at once to

$$(4.21) \quad c_2 = -3.$$

Applying condition (4.18),  $y'(0) = -1$ , to Equation (4.20), we obtain

$$-1 = e^0[(3c_1 - 4c_2)\sin 0 + (4c_1 + 3c_2)\cos 0]$$

which reduces to

$$(4.22) \quad 4c_1 + 3c_2 = -1.$$

Solving Equations (4.21) and (4.22) for the unknowns  $c_1$  and  $c_2$ , we find

$$\begin{cases} c_1 = 2 \\ c_2 = -3. \end{cases}$$

Replacing  $c_1$  and  $c_2$  in Equation (4.19) by these values, we obtain the unique solution of the given initial-value problem in the form

$$y = e^{3x}(2\sin 4x - 3\cos 4x).$$

We may write this in an alternate form by first multiplying and dividing by  $\sqrt{(2)^2 + (-3)^2} = \sqrt{13}$  to obtain

$$y = \sqrt{13}e^{3x} \left[ \frac{2}{\sqrt{13}}\sin 4x - \frac{3}{\sqrt{13}}\cos 4x \right].$$

From this we may express the solution in the alternate form

$$y = \sqrt{13}e^{3x}\sin(4x + \phi),$$

where the angle  $\phi$  is defined by the equations

$$\begin{cases} \sin \phi = -\frac{3}{\sqrt{13}} \\ \cos \phi = \frac{2}{\sqrt{13}}. \end{cases}$$

### Exercises

Find the general solution of each of the differential equations in Exercises 1 through 24.

1.  $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$

2.  $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0.$

3.  $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 5y = 0.$

4.  $3\frac{d^2y}{dx^2} - 14\frac{dy}{dx} - 5y = 0.$

5.  $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = 0.$