1.1 that Problem (B) has a solution. We are *not* saying that it does *not* have one. Theorem 1.1 simply gives no information one way or the other.

Exercises

1. Show that $y = 4e^{2x} + 2e^{-3x}$ is a solution of the initial-value problem

$$\begin{cases} \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0 \\ y(0) = 6 \\ y'(0) = 2. \end{cases}$$

Is $y = 2e^{2x} + 4e^{-3x}$ also a solution of this problem? Explain why or why not.

2. Given that the general solution of $\dfrac{dy}{dx} + y = 2xe^{-x}$ may be written $y = (x^2 + c)e^{-x}$, solve the following initial-value problems:

(A) $\begin{cases} \dfrac{dy}{dx} + y = 2xe^{-x} \\ y(0) = 2 \end{cases}$ (B) $\begin{cases} \dfrac{dy}{dx} + y = 2xe^{-x} \\ y(-1) = e + 3. \end{cases}$

3. Given that the general solution of $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 12y = 0$ may be written $y = c_1e^{4x} + c_2e^{-3x}$, solve the following initial-value problems:

(A) $\begin{cases} \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 12y = 0 \\ y(0) = 5 \\ y'(0) = 6. \end{cases}$ (B) $\begin{cases} \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 12y = 0 \\ y(0) = -2 \\ y'(0 = 6. \end{cases}$

4. The general solution of the differential equation $\dfrac{d^2y}{dx^2} + y = 0$ may be written in the form $y = c_1\sin x + c_2\cos x$. Using this information, show that boundary problems (A) and (B) below possess solutions but that Problem (C) does not.

(A) $\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(0) = 0 \\ y(\pi/2) = 1. \end{cases}$ (B) $\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(0) = 1 \\ y'(\pi/2) = -1. \end{cases}$ (C) $\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(0) = 0 \\ y(\pi) = 1. \end{cases}$

5. Given that the general solution of

$$x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} - 6y = 0$$

may be written $y = c_1x + c_2x^2 + c_3x^3$, solve the initial-value problem consisting of the above differential equation plus the three conditions

$$y(2) = 0, \quad y'(2) = 2, \quad y''(2) = 6.$$

6. Apply Theorem 1.1 to show that each of the following initial-value problems has a unique solution defined on some sufficiently small interval $|x - 1| \leq h$ about $x_0 = 1$.

(A) $\begin{cases} \dfrac{dy}{dx} = x^2 \sin y \\ y(1) = -2. \end{cases}$ (B) $\begin{cases} \dfrac{dy}{dx} = \dfrac{y^2}{x-2} \\ y(1) = 0. \end{cases}$

7. Consider the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = P(x)y^2 + Q(x)y \\ y(2) = 5, \end{cases}$$

where $P(x)$ and $Q(x)$ are both third degree polynomials in x. Has this problem a unique solution on some interval $|x - 2| \leq h$ about $x_0 = 2$? Explain why or why not.

SUGGESTED READING

Agnew (1)
Coddington (4)
Ford (5)
Kaplan (10)
Leighton (11)

2

FIRST-ORDER EQUATIONS FOR WHICH EXACT SOLUTIONS ARE OBTAINABLE

In this chapter we consider certain basic types of first-order equations for which exact solutions may be obtained by definite procedures. The purpose of this chapter is to gain ability to recognize these various types and to apply the corresponding methods of solutions. Of the types considered here, the so-called exact equations considered in Section 2.1 are in a sense the most basic, while the separable equations of Section 2.2 are in a sense the "easiest." The most important, from the point of view of applications, are the separable equations of Section 2.2 and the linear equations of Section 2.3. The remaining types are of various very special forms, and the corresponding methods of solution involve various devices. In short, we might describe this chapter as a collection of special "methods," "devices," "tricks," or "recipes," in descending order of kindness!

2.1 Exact Differential Equations and Integrating Factors

A. Standard Forms of First-Order Differential Equations

The first-order differential equations to be studied in this chapter may be expressed in either the form

$$\frac{dy}{dx} = f(x,y) \tag{2.1}$$

or the form

$$M(x,y)dx + N(x,y)dy = 0. \tag{2.2}$$

An equation in one of these forms may readily be written in the other form. For example, the equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x - y}$$

is of the form (2.1). It may be written

$$(x^2 + y^2)dx + (y - x)dy = 0,$$

which is of the form (2.2). The equation

$$(\sin x + y)dx + (x + 3y)dy = 0,$$

which is of the form (2.2), may be written in the form (2.1) as

$$\frac{dy}{dx} = -\frac{\sin x + y}{x + 3y}.$$

B. Exact Differential Equations

DEFINITION. Suppose $u = F(x,y)$, where F has continuous first partial derivatives. The *total differential du* is defined by the formula

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Example 2.1. Let u be given in terms of x and y by

$$u = xy^2 + 2x^3y$$

for all real (x,y). Then

$$\frac{\partial u}{\partial x} = y^2 + 6x^2y, \qquad \frac{\partial u}{\partial y} = 2xy + 2x^3,$$

and the total differential du is defined by

$$du = (y^2 + 6x^2y)dx + (2xy + 2x^3)dy.$$

DEFINITION. The expression

(2.3) $$Mdx + Ndy$$

is called an *exact differential* if there exists some u for which this expression is the total differential du. In other words, the expression (2.3) is an exact differential if there exists some u such that

$$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N.$$

If $Mdx + Ndy$ is an exact differential, then the differential equation

$$Mdx + Ndy = 0$$

is called an *exact differential equation.*

Example 2.2. The differential equation

(2.4) $$y^2dx + 2xydy = 0$$

is an exact differential equation, for the expression $y^2dx + 2xydy$ is an exact differential. Indeed, it is the total differential of u, where $u = xy^2$, since the coefficient of dx is $\frac{\partial u}{\partial x} = y^2$ and that of dy is $\frac{\partial u}{\partial y} = 2xy$. On the other hand, the more simple appearing equation

$$ydx + 2xdy = 0, \tag{2.5}$$

obtained from (2.4) by dividing through by y, is *not* exact.

In Example 2.2 we stated without hesitation that the differential equation (2.4) is exact but the differential equation (2.5) is not. In the case of the equation (2.4) we verified our assertion by actually exhibiting the function u of which the expression $y^2dx + 2xydy$ is the total differential. But in the case of equation (2.5), we did not back up our statement by showing that there is *no* u such that $ydx + 2xdy$ is its total differential. It is clear that we need a simple test to determine whether or not a given differential equation is exact. This is given by the following theorem.

THEOREM 2.1. Consider the differential equation

$$Mdx + Ndy = 0, \tag{2.6}$$

where M and N have continuous first partial derivatives.

(1) *If* the differential equation (2.6) is exact, *then*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{2.7}$$

(2) *Conversely, if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then the differential equation (2.6) is exact.

Proof. Part 1. If the differential equation (2.6) is exact, then $Mdx + Ndy$ is an exact differential. By definition of an exact differential, there exists a u such that

$$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N.$$

Then

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

But, using the continuity of the first partial derivatives of M and N, we have

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

and therefore

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Part 2. This being the converse of Part 1, we start with the hypothesis that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and set out to show that $Mdx + Ndy = 0$ is exact. This means that we must prove that there exists a u such that

$$\frac{\partial u}{\partial x} = M \tag{2.8}$$

and

$$\frac{\partial u}{\partial y} = N. \tag{2.9}$$

We can certainly find some u satisfying *either* (2.8) or (2.9), but what about *both?* Let us assume that u satisfies (2.8) and proceed.
Then

$$u = \int M\partial x + \phi(y), \tag{2.10}$$

where $\int M\partial x$ indicates a "partial integration" with respect to x, holding y constant, and ϕ is an arbitrary function of y only (this corresponds to a "constant of integration"). Differentiating (2.10) partially with respect to y, we obtain

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int M\partial x + \frac{d\phi}{dy}.$$

Now if (2.9) is to be satisfied, we must have

$$N = \frac{\partial}{\partial y}\int M\partial x + \frac{d\phi}{dy} \tag{2.11}$$

and hence

$$\frac{d\phi}{dy} = N - \frac{\partial}{\partial y}\int M\partial x.$$

Since ϕ is a function of y only, the derivative $\frac{d\phi}{dy}$ must also be independent of x. That is, in order for (2.11) to hold,

$$N - \frac{\partial}{\partial y}\int M\partial x \tag{2.12}$$

must be independent of x. Since

$$\frac{\partial}{\partial x}\left[N - \frac{\partial}{\partial y}\int M\partial x\right] = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x\partial y}\int M\partial x = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y\partial x}\int M\partial x = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

(since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ *by hypothesis*), we thus see that (2.12) *is* independent of x.

Thus we may write

$$\phi(y) = \int\left[N - \int\frac{\partial M}{\partial y}\partial x\right]dy.$$

Substituting this into Equation (2.10), we have

(2.13) $$u = \int M\partial x + \int \left[N - \int \frac{\partial M}{\partial y}\partial x \right] dy.$$

This u thus satisfies both (2.8) and (2.9), and so $Mdx + Ndy = 0$ *is* exact. *Q.E.D.*

Students well-versed in the terminology of higher mathematics will recognize that Theorem 2.1 may be stated in the following words: "A necessary and sufficient condition that equation (2.6) be exact is that condition (2.7) hold." For students not so well-versed, let us simply emphasize that condition (2.7), $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, is the criterion for exactness. If (2.7) holds, then (2.6) is exact: if (2.7) does *not* hold, then (2.6) is *not* exact.

Example 2.3. We apply the exactness criterion (2.7) to the Equations (2.4) and (2.5) introduced in Example 2.2. For the equation

(2.4) $$y^2dx + 2xydy = 0$$

we have $M = y^2$, $N = 2xy$, and $\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$. Thus the Equation (2.4) is exact. On the other hand, for the equation

(2.5) $$ydx + 2xdy = 0,$$

we have $M = y$, $N = 2x$, and $\frac{\partial M}{\partial y} = 1 \neq 2 = \frac{\partial N}{\partial x}$. Thus the Equation (2.5) is *not* exact.

Example 2.4. Consider the differential equation

$$(2x\sin y + y^3e^x)dx + (x^2\cos y + 3y^2e^x)dy = 0.$$

Here

$$M = 2x\sin y + y^3e^x,$$
$$N = x^2\cos y + 3y^2e^x,$$
$$\frac{\partial M}{\partial y} = 2x\cos y + 3y^2e^x = \frac{\partial N}{\partial x}.$$

Thus this differential equation is exact.

C. The Solution of Exact Differential Equations

Now that we have a test with which to determine exactness, let us proceed to solve exact differential equations. If the equation $Mdx + Ndy = 0$ is exact, then there exists some u such that $\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$. Then the equation may be written

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \quad \text{or simply} \quad du = 0.$$

The relation $u = c$ is obviously a solution of this, where c is an arbitrary constant. We summarize this observation in the following theorem.

THEOREM 2.2. The general solution of the exact differential equation $Mdx + Ndy = 0$ is given by $u = c$, where u is such that $\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$, and c is an arbitrary constant.

Referring to Theorem 2.1, we observe that u is given by formula (2.13). However, in solving exact differential equations it is neither necessary nor desirable to use this formula. Instead one obtains u either by proceeding as in the proof of Theorem 2.1, Part 2, or by the so-called "method of grouping," which will be explained in the following examples.

Example 2.5. $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$.

Our first duty is to determine whether or not the equation is exact. Here $M = 3x^2 + 4xy$ and $N = 2x^2 + 2y$, $\frac{\partial M}{\partial y} = 4x$ and $\frac{\partial N}{\partial x} = 4x$, and so the equation is exact. Thus we must find u such that $\frac{\partial u}{\partial x} = M = 3x^2 + 4xy$ and $\frac{\partial u}{\partial y} = N = 2x^2 + 2y$. From the first of these,

$$u = \int M\partial x + \phi(y) = \int (3x^2 + 4xy)\partial x + \phi(y) = x^3 + 2x^2y + \phi(y).$$

Then

$$\frac{\partial u}{\partial y} = 2x^2 + \frac{d\phi}{dy}.$$

But we must have $\frac{\partial u}{\partial y} = N = 2x^2 + 2y$.

Thus

$$2x^2 + 2y = 2x^2 + \frac{d\phi}{dy}$$

or

$$\frac{d\phi}{dy} = 2y.$$

Thus $\phi(y) = y^2 + c_0$, where c_0 is an arbitrary constant, and so

$$u = x^3 + 2x^2y + y^2 + c_0.$$

The general solution is $u = c_1$ or

$$x^3 + 2x^2y + y^2 + c_0 = c_1.$$

Absorbing together the nonessential constants c_0 and c_1 we may write our solution as

$$x^3 + 2x^2y + y^2 = c,$$

where $c = c_1 - c_0$ is an arbitrary constant.

The student will observe that there is no loss in generality by taking $c_0 = 0$ and writing $\phi(y) = y^2$. We now consider an alternate procedure.

Method of Grouping. We shall now solve the differential equation of this example by grouping the terms in such a way that its left member appears as the sum of certain exact differentials. We write the differential equation

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

in the form

$$3x^2dx + (4xydx + 2x^2dy) + 2ydy = 0.$$

We now recognize this as

$$d(x^3) + d(2x^2y) + d(y^2) = d(c),$$

where c is an arbitrary constant, or

$$d(x^3 + 2x^2y + y^2) = d(c).$$

From this we have at once

$$x^3 + 2x^2y + y^2 = c.$$

Clearly this procedure is much quicker, but it requires a good "working knowledge" of differentials and a certain amount of ingenuity to determine just how the terms should be grouped. The standard method may require more "work" and take longer, but it is perfectly straightforward. It is recommended for those who like to follow a pattern and for those who have a tendency to jump at conclusions.

Just to make certain that we have both procedures well in hand, we shall consider an initial-value problem involving an exact differential equation.

Example 2.6. Solve the initial-value problem

$$(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0,$$
$$y(0) = 2.$$

We first observe that the equation is exact:

$$\frac{\partial M}{\partial y} = -2x\sin y + 3x^2 = \frac{\partial N}{\partial x}.$$

"Standard" Method. We must find u such that

$$\frac{\partial u}{\partial x} = M = 2x\cos y + 3x^2y$$

and

$$\frac{\partial u}{\partial y} = N = x^3 - x^2\sin y - y.$$

Then

$$\begin{aligned} u &= \int M\partial x + \phi(y) \\ &= \int (2x\cos y + 3x^2y)\partial x + \phi(y) \\ &= x^2\cos y + x^3y + \phi(y), \\ \frac{\partial u}{\partial y} &= -x^2\sin y + x^3 + \frac{d\phi}{dy}. \end{aligned}$$

But also

$$\frac{\partial u}{\partial y} = N = x^3 - x^2\sin y - y$$

and so

$$\frac{d\phi}{dy} = -y$$

and hence

$$\phi(y) = -\frac{y^2}{2} + c_0.$$

Thus

$$u = x^2\cos y + x^3y - \frac{y^2}{2} + c_0,$$

and the general solution $u = c_1$ may be written

$$x^2\cos y + x^3y - \frac{y^2}{2} = c.$$

Applying the initial condition $y = 2$ when $x = 0$, we find $c = -2$. Thus the solution of the given initial-value problem is

$$x^2\cos y + x^3y - \frac{y^2}{2} = -2.$$

"*Method of Grouping.*" We group the terms as follows:

$$(2x\cos y dx - x^2\sin y dy) + (3x^2y dx + x^3 dy) - y dy = 0.$$

Thus we have

$$d(x^2\cos y) + d(x^3y) - d\left(\frac{y^2}{2}\right) = d(c);$$

and so

$$x^2\cos y + x^3y - \frac{y^2}{2} = c$$

is the general solution of the differential equation. Of course the initial condition $y(0) = 2$ again yields the particular solution already obtained.

D. Integrating Factors

Given the differential equation

$$M dx + N dy = 0,$$

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the equation is exact and we can obtain its solution by one of the procedures explained above. But if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the equation is *not* exact and the above procedures do not apply. What shall we do in such a case? Perhaps we can multiply the nonexact equation by some expression which will transform it into an equivalent exact equation. If so, we can proceed to solve the resulting exact equation by one of the above procedures. Let us consider again the equation

$$ydx + 2xdy = 0 \tag{2.5}$$

which was introduced in Example 2.2. In that example we observed that this equation is *not* exact. However, if we multiply it by y it is transformed into the equivalent equation

$$y^2dx + 2xydy = 0, \tag{2.4}$$

which, as we observed in Example 2.2, *is* exact. Since the resulting equation (2.4) (being exact) is integrable, we call y an *integrating factor* of the equation (2.5). In general, we have the following definition:

DEFINITION. Suppose that the differential equation

$$Mdx + Ndy = 0 \tag{2.14}$$

is *not* exact but that the differential equation

$$\mu Mdx + \mu Ndy = 0 \tag{2.15}$$

is exact, where $\mu = F(x,y)$ for a suitably chosen function F. Then μ is called an *integrating factor* of the differential equation (2.14).

Example 2.7. Consider the differential equation

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0. \tag{2.16}$$

This equation is of the form (2.14), where $M = 3y + 4xy^2$,

$$N = 2x + 3x^2y, \quad \frac{\partial M}{\partial y} = 3 + 8xy, \quad \text{and} \quad \frac{\partial N}{\partial x} = 2 + 6xy.$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, Equation (2.16) is *not* exact.

Let $\mu = x^2y$. Then the corresponding differential equation of the form (2.15) is

$$(3x^2y^2 + 4x^3y^3)dx + (2x^3y + 3x^4y^2)dy = 0.$$

This equation *is* exact, since

$$\frac{\partial(\mu M)}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial(\mu N)}{\partial x}.$$

Hence $\mu = x^2y$ is an integrating factor of Equation (2.16).

The question now arises: How is an integrating factor found? We shall not attempt to answer this question at this time. Instead we shall proceed to a study of the important classes of separable equations in Section 2.2 and linear equations in Section 2.3. We shall see that the former type always possess integrating factors which are perfectly obvious, while the latter always has integrating factors of a certain special form. We shall return to the question raised above in Section 2.4. Our object here has been merely to introduce the concept of an integrating factor.