(b) Show that the general solution of equation (A) can be written in the form

$$x = c\phi\left(\frac{y}{x}\right), \tag{C}$$

where c is an arbitrary constant.

(c) Use the result of (b) to show that the general solution (C) is also invariant under the transformation (B).

(d) Interpret geometrically the results proved in (a) and (c).

2.3 Linear Equations and Bernoulli Equations

A. Linear Equations

In Chapter I we gave a definition of linear as applied to differential equations in general; we now consider the linear ordinary differential equation of the first order.

DEFINITION. A first-order ordinary differential equation is called *linear* if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2.26}$$

For example, the equation

$$x\frac{dy}{dx} + (x + 1)y = x^3$$

is a first-order linear differential equation, for it can be written as

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = x^2,$$

which is of the form (2.26) with $P(x) = 1 + \frac{1}{x}$ and $Q(x) = x^2$.

Let us write the equation (2.26) in the form

$$[P(x)y - Q(x)]dx + dy = 0. \tag{2.27}$$

Equation (2.27) is of the form

$$Mdx + Ndy = 0,$$

where

$$M = P(x)y - Q(x) \quad \text{and} \quad N = 1.$$

Since

$$\frac{\partial M}{\partial y} = P(x) \quad \text{and} \quad \frac{\partial N}{\partial x} = 0,$$

the equation (2.27) is *not* exact unless $P(x) = 0$, in which case Equation (2.26) degenerates into a simple separable equation. However, the equation (2.27) possesses an integrating factor which depends on x only and may easily be found. Let us proceed to find it. Let us multiply equation (2.27) by $\mu(x)$, obtaining

(2.28) $$[\mu(x)P(x)y - \mu(x)Q(x)]dx + \mu(x)dy = 0.$$

By definition, $\mu(x)$ is an integrating factor of Equation (2.28) if and only if Equation (2.28) is exact; that is, if and only if

$$\frac{\partial}{\partial y}[\mu(x)P(x)y - \mu(x)Q(x)] = \frac{\partial}{\partial x}[\mu(x)].$$

This condition reduces to

$$\mu(x)P(x) = \frac{d}{dx}[\mu(x)]$$

or simply

(2.29) $$\mu P = \frac{d\mu}{dx}.$$

Equation (2.29) is a separable equation in the dependent variable μ and the independent variable x, where P is a known function of x. Separating the variables, we have

$$\frac{d\mu}{\mu} = Pdx.$$

Integrating, we obtain the particular solution

$$\ln|\mu| = \int Pdx$$

or

(2.30) $$\mu = e^{\int Pdx}.$$

Thus the linear equation (2.26) possesses an integrating factor of the form (2.30). Multiplying (2.26) by (2.30) gives

$$e^{\int Pdx}\frac{dy}{dx} + e^{\int Pdx}Py = Qe^{\int Pdx},$$

which is precisely

$$\frac{d}{dx}[e^{\int Pdx}y] = Qe^{\int Pdx}.$$

Integrating this we obtain the solution of Equation (2.26) in the form

$$e^{\int Pdx}y = \int e^{\int Pdx}Qdx + c,$$

where c is an arbitrary constant.
Summarizing this discussion, we have the following theorem:

THEOREM 2.4. The linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has an integrating factor of the form

$$e^{\int Pdx}.$$

The general solution of this equation is

$$y = e^{-\int P dx}\left[\int e^{\int P dx} Q dx + c\right].$$

We consider several examples.

Example 2.14.

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}. \tag{2.31}$$

Here

$$P(x) = \frac{2x+1}{x}$$

and hence an integrating factor is

$$e^{\int P(x)dx} = e^{\int\left(\frac{2x+1}{x}\right)dx} = e^{2x+\ln|x|} = e^{2x}e^{\ln|x|} = xe^{2x}.$$

Multiplying Equation (2.31) through by this integrating factor, we obtain

$$xe^{2x}\frac{dy}{dx} + e^{2x}(2x+1)y = x$$

or

$$\frac{d}{dx}[xe^{2x}y] = x.$$

Integrating, we obtain the solution

$$xe^{2x}y = \frac{x^2}{2} + c$$

or

$$y = \frac{1}{2}xe^{-2x} + \frac{c}{x}e^{-2x},$$

where c is an arbitrary constant.

Example 2.15. Solve the initial-value problem which consists of the differential equation

$$(x^2+1)\frac{dy}{dx} + 4xy = x \tag{2.32}$$

and the initial condition

$$y(2) = 1. \tag{2.33}$$

The differential equation (2.32) is not in the form (2.26). We therefore divide by $x^2 + 1$ to obtain

$$\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{x}{x^2+1}. \tag{2.34}$$

Equation (2.34) is in the standard form (2.26), where $P(x) = \frac{4x}{x^2 + 1}$. An integrating factor is

$$e^{\int P dx} = e^{\int \frac{4x dx}{x^2+1}} = e^{\ln(x^2+1)^2} = (x^2 + 1)^2.$$

Multiplying equation (2.34) through by this integrating factor, we have

$$(x^2 + 1)^2\frac{dy}{dx} + 4x(x^2 + 1)y = x(x^2 + 1)$$

or

$$\frac{d}{dx}[(x^2 + 1)^2 y] = x^3 + x.$$

We now integrate to obtain the general solution of equation (2.32) in the form

$$(x^2 + 1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + c.$$

Applying the initial condition (2.33), we have

$$25 = 6 + c.$$

Thus $c = 19$ and the solution of the initial-value problem under consideration is

$$(x^2 + 1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19.$$

Example 2.16. Consider the differential equation

(2.35) $$y^2 dx + (3xy - 1)dy = 0.$$

Solving for $\frac{dy}{dx}$, this becomes

$$\frac{dy}{dx} = \frac{y^2}{1 - 3xy},$$

which is clearly *not* linear in y. Also, equation (2.35) is *not* exact, separable, or homogeneous. It appears to be of a type which we have not yet encountered; but let us look a little closer. Observe that in a first-order differential equation the roles of dx and dy are interchangeable. Looking at equation (2.35) with this in mind, we write it as

$$\frac{dx}{dy} = \frac{1 - 3xy}{y^2}$$

or

(2.36) $$\frac{dx}{dy} + \frac{3}{y}x = \frac{1}{y^2}.$$

Now observe that equation (2.36) is of the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

and so is *linear in x*. Thus the theory developed in this section may be applied to the equation (2.36) merely by interchanging the roles played by x and y. Thus an integrating factor is

$$e^{\int P\,dy} = e^{\int \frac{3}{y} dy} = e^{\ln y^3} = y^3.$$

Multiplying (2.36) by y^3 we obtain

$$y^3 \frac{dx}{dy} + 3y^2 x = y$$

or

$$\frac{d}{dy}[y^3 x] = y.$$

Integrating, we find the solution in the form

$$y^3 x = \frac{y^2}{2} + c$$

or

$$x = \frac{1}{2y} + \frac{c}{y^3},$$

where c is an arbitrary constant.

B. Bernoulli Equations

We now consider a rather special type of equation which can be reduced to a linear equation by an appropriate transformation. This is the so-called Bernoulli equation.

DEFINITION. An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{2.37}$$

is called a *Bernoulli Differential Equation.*

We observe that if $n = 0$ or 1, then the Bernoulli Equation (2.37) is actually a linear equation and is therefore readily solvable as such. However, in the general case in which $n \neq 0$ or 1, this simple situation does not hold and we must proceed in a different manner. We now state and prove Theorem 2.5, which gives a method of solution in the general case.

THEOREM 2.5. Suppose $n \neq 0$ or 1. Then the transformation $v = y^{1-n}$ reduces the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{2.37}$$

to a linear equation in v.

Proof. We first multiply Equation (2.37) by y^{-n}, thereby expressing it in the equivalent form

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x). \tag{2.38}$$

If we let $v = y^{1-n}$, then $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ and Equation (2.38) transforms into

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

or, equivalently,

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

Letting

$$P_1(x) = (1-n)P(x)$$

and

$$Q_1(x) = (1-n)Q(x),$$

this may be written

$$\frac{dv}{dx} + P_1(x)v = Q_1(x),$$

which is linear in v. *Q.E.D.*

Example 2.17.

$$\frac{dy}{dx} + y = xy^3. \tag{2.39}$$

This is a Bernoulli differential equation, where $n = 3$. We first multiply the equation through by y^{-3}, thereby expressing it in the equivalent form

$$y^{-3}\frac{dy}{dx} + y^{-2} = x. \tag{2.40}$$

If we let $v = y^{1-n} = y^{-2}$, then $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$ and Equation (2.40) transforms into the linear equation

$$-\frac{1}{2}\frac{dv}{dx} + v = x.$$

Writing this linear equation in the standard form

$$\frac{dv}{dx} - 2v = -2x, \tag{2.41}$$

we see that an integrating factor for this equation is

$$e^{\int P\,dx} = e^{-\int 2\,dx} = e^{-2x}.$$

Multiplying (2.41) by e^{-2x}, we find

$$e^{-2x}\frac{dv}{dx} - 2e^{-2x}v = -2xe^{-2x}$$

or

$$\frac{d}{dx}[e^{-2x}v] = -2xe^{-2x}.$$

Integrating, we find

$$e^{-2x}v = \frac{1}{2}e^{-2x}(2x + 1) + c$$

or

$$v = x + \frac{1}{2} + ce^{2x}.$$

But

$$v = \frac{1}{y^2}.$$

Thus we obtain the solution of (2.39) in the form

$$\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}.$$

Exercises

Solve the given differential equations in Exercises 1 through 15.

1. $\frac{dy}{dx} + \frac{3y}{x} = 6x^2.$
2. $x^4\frac{dy}{dx} + 2x^3y = 1.$
3. $\frac{dx}{dt} + \frac{x}{t^2} = \frac{1}{t^2}.$
4. $(u^2 + 1)\frac{dv}{du} + 4uv = 3u.$
5. $x\frac{dy}{dx} + \frac{2x + 1}{x + 1}y = x - 1.$
6. $(x^2 + x - 2)\frac{dy}{dx} + 3(x + 1)y = x - 1.$
7. $xdy + (xy + y - 1)dx = 0.$
8. $ydx + (xy^2 + x - y)dy = 0.$
9. $\frac{dr}{d\theta} + r\tan\theta = \cos\theta.$