23. $\frac{d^2y}{dx^2} + 9y = e^{3x} + e^{-3x} + e^{3x}\sin 3x.$

24. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2e^x + 3xe^{2x} + 5x^2.$

25. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = xe^{2x} + x^2e^{3x}.$

26. $\frac{d^4y}{dx^4} + 3\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = x^2e^{-x} + 3e^{-1/2x}\cos\frac{\sqrt{3}}{2}x.$

27. $\frac{d^4y}{dx^4} - 16y = x^2\sin 2x + x^4e^{2x}.$

28. $\frac{d^6y}{dx^6} + 2\frac{d^5y}{dx^5} + 5\frac{d^4y}{dx^4} = x^3 + x^2e^{-x} + e^{-x}\sin 2x.$

29. $\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} - 4y = \cos^2 x - \cosh x.$

30. $\frac{d^4y}{dx^4} + 10\frac{d^2y}{dx^2} + 9y = \sin x \sin 2x.$

4.4 Variation of Parameters

A. The Method

While the process of carrying out the method of undetermined coefficients is actually quite straightforward (involving only techniques of college algebra and differentiation), the method applies in general to a rather small class of problems. For example, it would not apply to the apparently simple equation

$$\frac{d^2y}{dx^2} + y = \tan x.$$

We thus seek a method of finding a particular integral which applies in all cases (including variable coefficients) in which the complementary function is known. Such a method is the method of *variation of parameters*, which we now consider.

We shall develop this method in connection with the general second order linear differential equation with variable coefficients

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x). \tag{4.29}$$

Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0. \tag{4.30}$$

Then the complementary function of equation (4.29) is

$$c_1y_1 + c_2y_2,$$

where c_1 and c_2 are arbitrary constants. The procedure in the method of variation of parameters is to replace the arbitrary constants c_1 and c_2 in the complementary function by respective *functions* v_1 and v_2 which will be determined so that the resulting function

(4.31) $$v_1y_1 + v_2y_2$$

will be a particular integral of equation (4.29) (hence the name, *variation* of parameters).

We have at our disposal the *two functions* v_1 and v_2 with which to satisfy the *one condition* that (4.31) be a solution of (4.29). Since we have *two* functions but only *one* condition on them, we are thus free to impose a second condition, provided this second condition does not violate the first one. We shall see when and how to impose this additional condition as we proceed.

We thus assume a solution of the form (4.31) and write

(4.32) $$y_p = v_1y_1 + v_2y_2.$$

Differentiating (4.32), we have

(4.33) $$y_p' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2,$$

where we use primes to denote differentiations. At this point we impose the aforementioned second condition; we simplify y_p' by demanding that

(4.34) $$v_1'y_1 + v_2'y_2 = 0.$$

With this condition imposed, (4.33) reduces to

(4.35) $$y_p' = v_1y_1' + v_2y_2'.$$

Now differentiating (4.35), we obtain

(4.36) $$y_p'' = v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'.$$

We now impose the basic condition that (4.32) be a solution of Equation (4.29). Thus we substitute (4.32), (4.35), and (4.36) for y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$, respectively, in Equation (4.29) and obtain the identity

$$a_0[v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'] + a_1[v_1y_1' + v_2y_2'] + a_2[v_1y_1 + v_2y_2] = F.$$

This can be written as

(4.37) $$v_1[a_0y_1'' + a_1y_1' + a_2y_1] + v_2[a_0y_2'' + a_1y_2' + a_2y_2] + a_0[v_1'y_1' + v_2'y_2'] = F.$$

Since y_1 and y_2 are solutions of the corresponding homogeneous differential equation (4.30), the expressions in the first two brackets in (4.37) are identically zero. This leaves merely

(4.38) $$v_1'y_1' + v_2'y_2' = \frac{F}{a_0}.$$

This is actually what the basic condition demands. Thus the two imposed conditions require that the functions v_1 and v_2 be chosen such that the system of equations

(4.39) $$\begin{cases} y_1v_1' + y_2v_2' = 0 \\ y_1'v_1' + y_2'v_2' = \dfrac{F}{a_0} \end{cases}$$

is satisfied. The determinant of coefficients of this system is precisely

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous differential equation (4.30), we know that $W(y_1, y_2) \neq 0$. Hence the system (4.39) has a unique solution. Actually solving this system we obtain

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \dfrac{F}{a_0} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{Fy_2}{a_0 W(y_1, y_2)}$$

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \dfrac{F}{a_0} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{Fy_1}{a_0 W(y_1, y_2)}.$$

Thus we obtain the functions v_1 and v_2 given by

(4.40)

$$v_1(x) = -\int^x \frac{F(t)y_2(t)dt}{a_0(t)W[y_1(t), y_2(t)]}$$

$$v_2(x) = \int^x \frac{F(t)y_1(t)dt}{a_0(t)W[y_1(t), y_2(t)]}.$$

Therefore a particular integral of equation (4.29) is

$$y_p = v_1y_1 + v_2y_2,$$

where v_1 and v_2 are defined by (4.40).

B. Examples

Example 4.34.

(4.41) $$\frac{d^2y}{dx^2} + y = \tan x.$$

The complementary function is

$$y_c = c_1\sin x + c_2\cos x.$$

We assume

(4.42) $$y_p = v_1\sin x + v_2\cos x,$$

where the functions v_1 and v_2 will be determined such that this is a particular integral of the differential equation (4.41). Then

$$y_p' = v_1\cos x - v_2\sin x + v_1'\sin x + v_2'\cos x.$$

We impose the condition

(4.43) $$v_1'\sin x + v_2'\cos x = 0,$$

leaving

$$y_p' = v_1\cos x - v_2\sin x.$$

From this

(4.44) $$y_p'' = -v_1\sin x - v_2\cos x + v_1'\cos x - v_2'\sin x.$$

Substituting (4.42) and (4.44) into (4.41) we obtain

(4.45) $$v_1'\cos x - v_2'\sin x = \tan x.$$

Thus we have the two equations (4.43) and (4.45) from which to determine v_1', v_2':

$$\begin{cases} v_1'\sin x + v_2'\cos x = 0 \\ v_1'\cos x - v_2'\sin x = \tan x. \end{cases}$$

Solving we find:

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \tan x}{-1} = \sin x,$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \tan x}{-1} = \frac{-\sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x.$$

Integrating we find:

(4.46) $$\begin{cases} v_1 = -\cos x + c_3, \\ v_2 = \sin x - \ln|\sec x + \tan x| + c_4. \end{cases}$$

Substituting (4.46) into (4.42) we have

$$\begin{aligned} y_p &= [-\cos x + c_3]\sin x + [\sin x - \ln|\sec x + \tan x| + c_4]\cos x \\ &= -\sin x\cos x + c_3\sin x + \sin x\cos x \\ &\quad - \ln|\sec x + \tan x|(\cos x) + c_4\cos x \\ &= c_3\sin x + c_4\cos x - (\cos x)[\ln|\sec x + \tan x|]. \end{aligned}$$

Since a particular integral is a solution free of arbitrary constants, we may assign any particular values A and B to c_3 and c_4, respectively, and the result will be the particular integral

$$A\sin x + B\cos x - (\cos x)[\ln|\sec x + \tan x|].$$

Thus $y = y_c + y_p$ becomes:

$$y = c_1\sin x + c_2\cos x + A\sin x + B\cos x - (\cos x)\ln|\sec x + \tan x|$$

which we may write as

$$y = C_1 \sin x + C_2 \cos x - (\cos x) \ln |\sec x + \tan x|,$$

where $C_1 = c_1 + A$, $C_2 = c_2 + B$.

Thus we see that we might as well have chosen the constants c_3 and c_4 both equal to 0 in (4.46), for essentially the same result, $y = c_1 \sin x + c_2 \cos x - (\cos x) \ln |\sec x + \tan x|$, would have been obtained. This is the general solution of the differential equation (4.41).

The method of variation of parameters extends to higher order linear equations. We now illustrate the extension to a third-order equation in Example 4.35, although we hasten to point out that the equation of this example can be solved more readily by the method of undetermined coefficients.

Example 4.35.

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^x. \tag{4.47}$$

The complementary function is

$$y_c = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

We assume as a particular integral

$$y_p = v_1e^x + v_2e^{2x} + v_3e^{3x}. \tag{4.48}$$

Since we have *three* functions v_1, v_2, v_3 at our disposal in this case, we can apply three conditions. We have:

$$y_p' = v_1e^x + 2v_2e^{2x} + 3v_3e^{3x} + v_1'e^x + v_2'e^{2x} + v_3'e^{3x}.$$

Proceeding in a manner analogous to that of the second-order case, we impose the condition

$$v_1'e^x + v_2'e^{2x} + v_3'e^{3x} = 0, \tag{4.49}$$

leaving

$$y_p' = v_1e^x + 2v_2e^{2x} + 3v_3e^{3x}. \tag{4.50}$$

Then

$$y_p'' = v_1e^x + 4v_2e^{2x} + 9v_3e^{3x} + v_1'e^x + 2v_2'e^{2x} + 3v_3'e^{3x}.$$

We now impose the condition

$$v_1'e^x + 2v_2'e^{2x} + 3v_3'e^{3x} = 0, \tag{4.51}$$

leaving

$$y_p'' = v_1e^x + 4v_2e^{2x} + 9v_3e^{3x}. \tag{4.52}$$

From this,

$$y_p''' = v_1e^x + 8v_2e^{2x} + 27v_3e^{3x} + v_1'e^x + 4v_2'e^{2x} + 9v_3'e^{3x}. \tag{4.53}$$

We substitute (4.48), (4.50), (4.52), and (4.53) into the differential equation (4.47), obtaining:

$$\begin{aligned} v_1e^x + 8v_2e^{2x} + 27v_3e^{3x} + v_1'e^x + 4v_2'e^{2x} + 9v_3'e^{3x} \\ - 6v_1e^x - 24v_2e^{2x} - 54v_3e^{3x} \\ + 11v_1e^x + 22v_2e^{2x} + 33v_3e^{3x} \\ - 6v_1e^x - 6v_2e^{2x} - 6v_3e^{3x} = e^x \end{aligned}$$

or

$$v'e^x + 4v_2'e^{2x} + 9v_3'e^{3x} = e^x. \tag{4.54}$$

Thus we have the three equations (4.49), (4.51), (4.54) from which to determine v_1', v_2', v_3':

$$\begin{cases} v_1'e^x + v_2'e^{2x} + v_3'e^{3x} = 0 \\ v_1'e^x + 2v_2'e^{2x} + 3v_3'e^{3x} = 0 \\ v_1'e^x + 4v_2'e^{2x} + 9v_3'e^{3x} = e^x. \end{cases}$$

Solving, we find:

$$v_1' = \frac{\begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}} = \frac{e^{6x}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}}{e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}} = \frac{1}{2},$$

$$v_2' = \frac{\begin{vmatrix} e^x & 0 & e^{3x} \\ e^x & 0 & 3e^{3x} \\ e^x & e^x & 9e^{3x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}} = \frac{-e^{5x}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{2e^{6x}} = -e^{-x},$$

$$v_3' = \frac{\begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & e^x \end{vmatrix}}{\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}} = \frac{e^{4x}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{2e^{6x}} = \frac{1}{2}e^{-2x}.$$

We now integrate, choosing all the constants of integration to be zero (as the previous example showed was possible). We find:

$$v_1 = \frac{1}{2}x, \quad v_2 = e^{-x}, \quad v_3 = -\frac{1}{4}e^{-2x}.$$

Thus

$$y_p = \frac{1}{2}xe^x + e^{-x}e^{2x} - \frac{1}{4}e^{-2x}e^{3x} = \frac{1}{2}xe^x + \frac{3}{4}e^x,$$

and so

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + c_3e^{3x} + \frac{1}{2}xe^x + \frac{3}{4}e^x$$

$$= c_1'e^x + c_2e^{2x} + c_3e^{3x} + \frac{1}{2}xe^x,$$

where $c_1' = c_1 + \frac{3}{4}$.

In Examples 4.34 and 4.35 the coefficients in the differential equation were constants. The general discussion at the beginning of this section shows that the method applies equally well to linear differential equations with variable coefficients, once the complementary function y_c is known. We now illustrate its application to such an equation in Example 4.36.

Example 4.36.

$$(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2 + 1)^2. \tag{4.55}$$

In Example 4.15 we solved the corresponding homogeneous equation

$$(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

From the results of that example, we see that the complementary function of equation (4.55) is

$$y_c = c_1x + c_2(x^2 - 1).$$

To find a particular integral of equation (4.55), we therefore let

$$y_p = v_1x + v_2(x^2 - 1), \tag{4.56}$$

where v_1 and v_2 are functions of x. Then

$$y_p' = v_1 \cdot 1 + v_2 \cdot 2x + v_1'x + v_2'(x^2 - 1).$$

We impose the condition

$$v_1'x + v_2'(x^2 - 1) = 0, \tag{4.57}$$

leaving

$$y_p' = v_1 \cdot 1 + v_2 \cdot 2x. \tag{4.58}$$

From this, we find

$$y_p'' = v_1' + 2v_2 + v_2' \cdot 2x. \tag{4.59}$$

Substituting (4.56), (4.58), and (4.59) into (4.55) we obtain

$$(x^2 + 1)(v_1' + 2v_2 + 2xv_2') - 2x(v_1 + 2xv_2) + 2[v_1x + v_2(x^2 - 1)] = 6(x^2 + 1)^2$$

or

$$(x^2 + 1)(v_1' + 2xv_2') = 6(x^2 + 1)^2. \tag{4.60}$$

Thus we have the two equations (4.57) and (4.60) from which to determine v_1' and v_2'; that is, v_1' and v_2' satisfy the system

$$\begin{cases} v_1'x + v_2'(x^2 - 1) = 0, \\ v_1' + v_2'(2x) = 6(x^2 + 1). \end{cases}$$

Solving this system, we find

$$v_1' = \frac{\begin{vmatrix} 0 & x^2-1 \\ 6(x^2+1) & 2x \end{vmatrix}}{\begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix}} = \frac{-6(x^2+1)(x^2-1)}{x^2+1} = -6(x^2-1)$$

$$v_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & 6(x^2+1) \end{vmatrix}}{\begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix}} = \frac{6x(x^2+1)}{x^2+1} = 6x.$$

Integrating, we obtain

$$\text{(4.61)} \qquad \begin{cases} v_1 = -2x^3 + 6x, \\ v_2 = 3x^2, \end{cases}$$

where we have chosen both constants of integration to be zero. Substituting (4.61) into (4.56), we have

$$\begin{aligned} y_p &= (-2x^3 + 6x)x + 3x^2(x^2 - 1) \\ &= x^4 + 3x^2. \end{aligned}$$

Therefore the general solution of equation (4.55) may be expressed in the form

$$\begin{aligned} y &= y_c + y_p \\ &= c_1x + c_2(x^2 - 1) + x^4 + 3x^2. \end{aligned}$$

C. The Superposition Principle for Particular Integrals

If the nonhomogeneous member of a linear differential equation is expressed as a linear combination of two or more functions, the following theorem may often be used to advantage in finding a particular integral.

THEOREM 4.12.

Hypothesis.

(1) Let f be a particular integral of

$$\text{(4.62)} \qquad a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x).$$

(2) Let g be a particular integral of

$$\text{(4.63)} \qquad a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = G(x).$$

Conclusion. Then $k_1f + k_2g$ is a particular integral of

$$\text{(4.64)} \quad a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = k_1F(x) + k_2G(x),$$

where k_1 and k_2 are constants.

Let us apply this theorem to an equation of the form (4.64), where the coefficients $a_0, \ldots, a_n$ are constants. Suppose that F is a simple UC function but that G is not. Then we could *not* apply the method of undetermined coefficients to find a particular integral of Equation (4.64). However, we could find a particular integral of equation (4.62) by the method of undetermined coefficients, and then use variation of parameters to find a particular integral of Equation (4.63). Then applying Theorem 4.12, the appropriate linear combination of these two particular integrals, found by different methods, is a particular integral of Equation (4.64).

Example 4.37. Find a particular integral of

$$\frac{d^2y}{dx^2} + y = 3e^x + 5\tan x. \tag{4.65}$$

We consider the two equations

$$\frac{d^2y}{dx^2} + y = e^x \tag{4.66}$$

and

$$\frac{d^2y}{dx^2} + y = \tan x. \tag{4.67}$$

Since the function defined by e^x is a UC function, a particular integral of Equation (4.66) may be found by the method of undetermined coefficients. Letting $y_p = Ae^x$, we find at once that $A = \frac{1}{2}$; hence a particular integral of (4.66) is

$$y_p = \frac{1}{2}e^x.$$

Since the function defined by $\tan x$ is *not* a UC function, we turn to variation of parameters to find a particular integral of Equation (4.67). We have already solved this problem in Example 4.34; we found there that a particular integral of (4.67) is

$$y_p = -(\cos x)\ln|\sec x + \tan x|.$$

Thus, applying Theorem 4.12, a particular integral of Equation (4.65) is

$$y_p = \frac{3}{2}e^x - 5(\cos x)\ln|\sec x + \tan x|.$$

Exercises

Find the general solution of each of the differential equations in Exercises 1 through 13.

1. $\dfrac{d^2y}{dx^2} + y = \cot x.$

2. $\dfrac{d^2y}{dx^2} + y = \tan^2 x.$

3. $\dfrac{d^2y}{dx^2} + y = \sec x.$