

Transmission Lines Physics 623

Murray Thompson

Sept. 1, 1999

Contents

| | | |
|------|--|----|
| 1 | Introduction | 2 |
| 2 | Equations for a “lossless” Transmission Line | 2 |
| 3 | The Voltage Solution | 5 |
| 4 | The Current Solution | 5 |
| 5 | The “Characteristic Impedance Z_0 ” | 6 |
| 6 | Speed u of Signals | 6 |
| 7 | Impedances of Actual Cables | 6 |
| 8 | Eleven Examples | 10 |
| 9 | Capacitive Termination | 16 |
| 10 | Types of Transmission Lines | 22 |
| 11 | Imperfections of Transmission Lines | 23 |
| 11.1 | “Remedies” for Signal Loss | 24 |

| | |
|---|-----------|
| <i>CONTENTS</i> | 2 |
| 12 Equations for a “Lossy” Transmission Line | 25 |
| 13 Lossy Line with No Reflections | 29 |
| 14 Attenuation if line is slightly lossy | 29 |
| 15 Characteristic Impedance of a Lossy Transmission Line | 30 |
| 16 Heavyside Distortionless Lines | 30 |
| 17 Three Examples | 31 |
| 18 Attenuation at DC and Low Frequencies | 31 |
| 19 Attenuation at Higher Frequencies | 33 |
| 19.1 Skin Effect Loss | 33 |
| 19.2 Dielectric Loss | 35 |
| 19.3 Radiation Loss | 35 |
| 19.4 Actual Attenuation in Cables | 35 |

1 Introduction

A Transmission line is a pair of conductors which have a cross which remains constant with distance. For example, a coaxial cable transmission line has a cross section of a central rod and an outer concentric cylinder.

Similarly a twisted pair transmission line has two conducting rods or wires which slowly wind around each other. A cross section made at any distance along the line is the same as a cross section made at any other point on the line.

We want to understand the voltage - Current relationships of transmission lines.

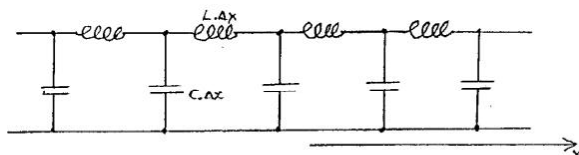
2 Equations for a “lossless” Transmission Line

A transmission line has a distributed inductance on each line and a distributed capacitance between the two conductors. We will consider the line to have zero series resistance and the insulator to have infinite resistance (a zero conductance or perfect insulator). We will consider a “Lossy” line later in section 12 on page 25.

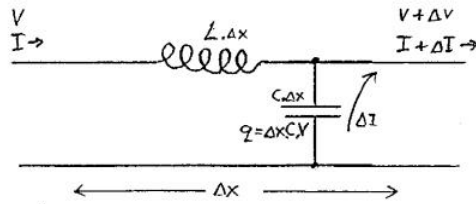
Define L to be the inductance/unit length and C to be the capacitance/unit length.

Consider a transmission line to be a pair of conductors divided into a number of cells with each cell having a small inductance in one line and having small capacitance to the other line.

In the limit of these cells being very small, they can represent a distributed inductance with distributed capacitance to the other conductor.



Consider one such cell corresponding to the components between position x and position $x + \Delta x$ along the transmission line.



The small series inductance is $L \cdot \Delta x$ and the small parallel capacitance is $C \cdot \Delta x$.

Define the voltage and current to the right on the left side to be V and I . Define the voltage and current to the right on the right side to be $V + \Delta V$ and $I + \Delta I$.

We now can get two equations.

1. The current increment ΔI between the left and right ends of the cell is discharging the capacitance in the cell.

The charge on the cell's capacitance = capacitance x voltage = $C \cdot \Delta x \cdot V$ and so the current leaving the capacitance to provide ΔI must be;

$$\Delta I = -\frac{\partial}{\partial t}(\text{Charge}) = -\frac{\partial}{\partial t}(C \cdot \Delta x \cdot V)$$

The minus sign is due to the current leaving the capacitor.

$$\Delta I = -C \cdot \Delta x \cdot \frac{\partial V}{\partial t}$$

$$\frac{\Delta I}{\Delta x} = -C \cdot \frac{\partial V}{\partial t}$$

Note the minus sign.

2. The voltage increment ΔV between the left and right ends of the cell is due to the changing current through the cell's inductance. (Lenz's Law)

$$\Delta V = -\text{Inductance} \cdot \frac{\partial I}{\partial t} = -\Delta x \cdot L \cdot \frac{\partial I}{\partial t}$$

$$\frac{\Delta V}{\Delta x} = -L \cdot \frac{\partial I}{\partial t}$$

Now take the limit of the cell being made very small so that the inductance and capacitance are uniformly distributed. The two equations then become

$$\frac{\partial I}{\partial x} = -C \cdot \frac{\partial V}{\partial t} \quad \text{Equation 1.}$$

$$\frac{\partial V}{\partial x} = -L \cdot \frac{\partial I}{\partial t} \quad \text{Equation 2.}$$

Remember that L and C are the inductance/unit length measured, in Henries/meter and are the capacitance/unit length measured in Farads/meter.

Differentiate equation 2 with respect to the distance x .

$$\frac{\partial}{\partial x} \left(\frac{\partial V}{\partial x} \right) = -L \cdot \frac{\partial}{\partial x} \left(\frac{\partial I}{\partial t} \right)$$

$$\frac{\partial^2 V}{\partial x^2} = -L \cdot \frac{\partial}{\partial x} \left(\frac{\partial I}{\partial t} \right)$$

x and t are independent variables and so the order of the partials can be changed.

$$\frac{\partial^2 V}{\partial x^2} = -L \cdot \frac{\partial}{\partial t} \left(\frac{\partial I}{\partial x} \right)$$

Now substitute for $\frac{\partial I}{\partial x}$ from equation 1 above

$$\frac{\partial^2 V}{\partial x^2} = -L \cdot \frac{\partial}{\partial t} \left(-C \cdot \frac{\partial V}{\partial t} \right)$$

$$\frac{\partial^2 V}{\partial x^2} = LC \cdot \frac{\partial^2 V}{\partial t^2} \quad \text{Equation 3}$$

This is usually called the Transmission Line Differential Equation.

Notes

- L and C are NOT just the inductance and the capacitance. They are both measured per unit length.
- The Transmission Line Differential Equation 3 above does NOT have a minus sign.

The Transmission Line Differential Equation 3 above is a normal 1 dimensional wave equation and is very similar to other wave equations in physics. From experience with such wave equations, we can try the normal solution of the form

$$V = V(s)$$

where s is a new variable $s = x + ut$. Substituting this into the two sides of the Transmission Line Differential Equation 3 above we get the two sides being

$$\frac{\partial^2 V}{\partial x^2} \text{ and } \frac{1}{u^2} \cdot \frac{\partial^2 V}{\partial t^2}$$

Thus the form $V(x + ut)$ can satisfy the Transmission Line Differential Equation 3 if and only if

$$\frac{1}{u^2} = LC \quad \text{Equation 4.}$$

Both roots of this satisfy the Equation 3.

$$u = \pm \frac{1}{\sqrt{LC}}$$

The two roots give slightly different solutions and so, since the equation 3 is linear, any linear combination of the two solutions is a valid solution. Define u as the positive root

$$u = + \frac{1}{\sqrt{LC}} \quad \text{Equation 5.}$$

3 The Voltage Solution

Thus, the general solution for the voltage is the linear combination.

$$V = f(x - ut) + g(x + ut) \quad \text{Equation 6.}$$

Where $f()$ and $g()$ are arbitrary single valued functions which can be very different.

1. $f(x - ut)$ describes a wave propagating with no change in shape towards $x = +\infty$.
2. $g(x + ut)$ describes a wave propagating with no change in shape towards $x = -\infty$.

4 The Current Solution

Consider one of the waves such as the “forward wave” propagating towards $x = +\infty$.

$$V = V(x - ut)$$

From this we can show, by differentiating, that:

$$\begin{aligned} -u \frac{\partial V}{\partial x} &= \frac{\partial V}{\partial t} \\ \frac{\partial V}{\partial x} &= -\frac{1}{u} \cdot \frac{\partial V}{\partial t} \end{aligned} \quad \text{Equation 7.}$$

Also from equation 2. above

$$\frac{\partial V}{\partial x} = -L \cdot \frac{\partial I}{\partial t} \quad \text{Equation 2.}$$

Equation 2 and equation 7 will have a common solution only if the two right hand sides are the same

$$\frac{1}{u} \cdot \frac{\partial V}{\partial t} = L \cdot \frac{\partial I}{\partial t}$$

$$V = uL \cdot I$$

This can be rewritten using $u = \frac{1}{\sqrt{LC}}$ from equation 5.

$$V = \sqrt{\frac{L}{C}} \cdot I$$

and

$$\frac{V}{I} = \sqrt{\frac{L}{C}}$$

and the current I of the forward wave is

$$I = V / \sqrt{\frac{L}{C}}$$

and, similarly for the backward wave

$$V = -\sqrt{\frac{L}{C}} \cdot I$$

$$\frac{V}{I} = -\sqrt{\frac{L}{C}}$$

and the current I of the backward wave is

$$I = -V/\sqrt{\frac{L}{C}}$$

Thus the general solution for both waves for the current I is

$$I = (f(x - ut) - g(x + ut))/\sqrt{\frac{L}{C}} \quad \text{Equation 7}$$

which can be compared with the earlier equation for the voltage

$$V = f(x - ut) + g(x + ut) \quad \text{Equation 6.}$$

5 The “Characteristic Impedance Z_0 ”

Define the “Characteristic Impedance Z_0 ” as the magnitude of the instantaneous ratio for either the forward wave or backward wave. For the forward wave:

$$Z_0 = \left| \frac{\text{Voltage}}{\text{Current}} \right| = \left| \frac{V}{I} \right| = \left| \sqrt{\frac{L}{C}} \right|$$

For the backward wave:

$$Z_0 = \left| \frac{\text{Voltage}}{\text{Current}} \right| = \left| \frac{V}{I} \right| = \left| \sqrt{\frac{L}{C}} \right|$$

With this definition of Z_0 the voltage and current equations can be written:

$$V = f(x - ut) + g(x + ut) \quad \text{Equation 6.}$$

$$I = \frac{f(x-ut)}{Z_0} - \frac{g(x+ut)}{Z_0} \quad \text{Equation 8.}$$

6 Speed u of Signals

The Inductance per unit length L and Capacitance per unit length C can be calculated from Electromagnetic Theory. The formulae depend upon the cross sectional shape of the conductors.

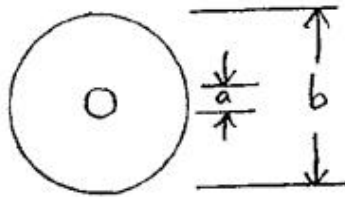
7 Impedances of Actual Cables

• Coaxial Cable

EM theory says that a Coaxial Cable with inner rod having diameter a and outer tube having diameter b has

$$C = \frac{2\pi\epsilon}{\ln(\frac{b}{a})} \text{ and}$$

$$L = \frac{\mu}{2\pi} \ln(\frac{b}{a}).$$



From these, u and Z_0 can be

obtained

$$u = \frac{1}{\sqrt{\mu\epsilon}} \text{ and}$$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{4\pi^2\epsilon}} \ln(\frac{b}{a}) \approx \frac{60}{\epsilon_r} \ln(\frac{b}{a})$$

For a vacuum, of course,

$$u_{vac} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \text{Velocity of light} \text{ and}$$

for polyethylene, ϵ_r is about 4, and the speed is halved

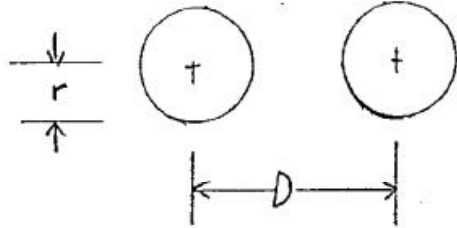
$$u_{polyethylene} = \frac{\text{Velocity of light}}{2}$$

• Parallel Cylinders

EM theory says that a pair of parallel conducting cylinders, rods or wires, with rod radius r and center to center separation D have

$$C = \frac{\pi\epsilon}{\ln(\frac{D}{r})}$$

$$L = \frac{\mu}{\pi} \ln(\frac{D}{r})$$



From these, u and Z_0 can be obtained $u = \frac{1}{\sqrt{\mu\epsilon}}$ and $Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{4\pi^2\epsilon}} \ln\left(\frac{D}{r}\right) \approx \frac{240}{\epsilon_r} \ln\left(\frac{D}{r}\right)$

Note that

1. in both cases the speed u is the same and depends only upon the medium. In fact, of course, for all possible cross sectional shapes, the speed is the same $u = \frac{1}{\sqrt{\mu\epsilon}}$ and if the medium is a vacuum, $u = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$ the speed of the voltage/current signal on the cable is the speed of light.
2. in both cases, the speed and the Characteristic Impedance depend upon logarithms of the ratio of two distances in the cross.
 - thus a big transmission line can have the same impedance as a small transmission line if one is scaled in proportion from the other.
 - For most lines it is not practical to vary the ratios $\frac{b}{a}$ and $\frac{D}{r}$ much more than about 2.0/1 up to 10/1. Since the $\ln\left(\frac{2}{1}\right) \approx 0.69$ and $\ln\left(\frac{10}{1}\right) \approx 2.3$ the range of impedances is normally within a moderate range (about 20 ohm to 200 ohm).

Some typical values can be found for $Z_0 = 50 \text{ ohm}$ cable with a polyethylene dielectric and speed $u = \text{half of light} = 1.5 \times 10^8 \text{ m/s}$. Use the speed equation from equation 5 of section 2 on page 4 and the characteristic impedance equation from section 5 on page 6.

$$u = \frac{1}{\sqrt{LC}} \text{ and } Z_0 = \sqrt{\frac{L}{C}}$$

By multiplying and dividing these equations, we can get L and C

$$L = \frac{Z_0}{u} \text{ and } C = \frac{1}{Z_0 u}$$

For the values assumed for Z_0 and u

$$\begin{aligned} L &= \frac{50 \text{ ohm}}{1.5 \times 10^8 \text{ m/s}} = 3.333 \times 10^{-7} \text{ Henry/meter} \\ &= 333 \times 10^{-9} \text{ Henry/meter} = 333 \text{ nanoHenry/meter} \end{aligned}$$

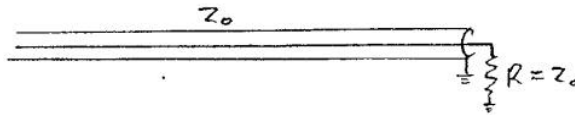
$$C = \frac{1}{50 \text{ ohm} \times 1.5 \times 10^8 \text{ m/s}} = \frac{1}{75 \times 10^8} \text{ Farad/m}$$
$$= 1.333 \times 10^{10} \text{ Farad/m} = 133.3 \text{ pF/m}$$

Thus a foot of RG58 cable with $Z=50$ ohm and u =half of light has a capacitance of $\approx 0.305\text{m/ft} \times 1.333\text{pF/m} = 40 \text{ pF}$.

8 Eleven Examples

To get a better feeling for the significance of the above, consider 11 examples.

1. "Receiver Termination or Parallel Termination with $R = Z_0$ "



Consider a transmission line which is terminated by a resistor, to ground, with resistance R which is **equal to** to characteristic impedance Z_0 . A signal is introduced by a module to the left (at negative x).

The general solution for V and I , from before, is

$$I = (f(x - ut) - g(x + ut))/Z_0 \quad \text{Equation 7}$$

$$V = f(x - ut) + g(x + ut) \quad \text{Equation 6.}$$

At $x = 0$ the V and I must obey these 2 equations and ALSO obey the equation (Ohm's Law) for the resistor.

$$V = IR = IZ_0$$

These can only be consistent if the functions $g() = 0$ so that

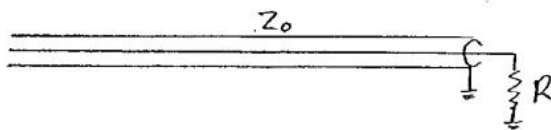
$$I = (f(x - ut))/Z_0$$

$$V = f(x - ut)$$

Thus, while there can be a forward wave, there cannot be a backward wave. Thus and forward wave cannot produce a backward wave. There are NO REFLECTIONS.

This example with a resistor $R = Z_0$ is sometimes called using parallel termination.

2. Any Resistance Value.



Now consider a similar signal introduced from the left but with a general value of R instead of requiring $R = Z_0$.

$$V = f + g$$

$$I = \frac{f}{Z_0} - \frac{g}{Z_0}$$

If $V = IR$ then

$$V = f + g = \left(\frac{f}{Z_0} - \frac{g}{Z_0}\right) \cdot R$$

$$f + g = f \frac{R}{Z_0} - g \frac{R}{Z_0}$$

$$f \left(1 - \frac{R}{Z_0}\right) = -g \left(1 + \frac{R}{Z_0}\right)$$

$$g = f \cdot \left(\frac{R - Z_0}{R + Z_0}\right)$$

Thus the forward wave causes a backward wave. The backward wave is, in general, smaller and we call

$$\mathcal{R} = \frac{g}{f} = \left(\frac{R - Z_0}{R + Z_0}\right) \text{ the "voltage reflection coefficient"}$$

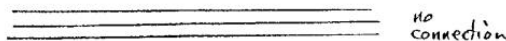
checks

- If the cable has $Z_0 = 50 \text{ ohm}$ and the resistor has $R = 100 \text{ ohm}$ then

$$\mathcal{R} = \frac{g}{f} = \frac{100 - 50}{100 + 50} = \frac{1}{3}$$

- If $R = Z_0$, the Voltage Reflection Coefficient $\mathcal{R} = \frac{g}{f} = \left(\frac{R - Z_0}{R + Z_0}\right) = \frac{0}{2R} = 0$ as in the previous example.

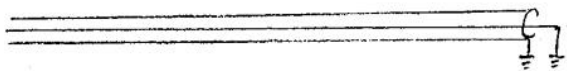
3. Open Circuit "Termination" $R = \infty$



If the cable is open circuited (ie has no resistor) then $R = \infty \text{ ohm}$ and using some sloppy algebra $\mathcal{R} = \frac{g}{f} = \left(\frac{\infty - Z_0}{\infty + Z_0}\right) = \frac{\infty}{\infty} = +1$

The backward wave will have the same size and shape as the incident right forward wave.

4. Short Circuit "Termination" $R = 0$



If the cable is short circuited (ie has zero resistance) then $R = 0 \text{ ohm}$ and $\mathcal{R} = \frac{g}{f} = \left(\frac{0 - Z_0}{0 + Z_0}\right) = \left(-\frac{Z_0}{Z_0}\right) = -1$

The backward wave will have the same size and shape as the incident right forward wave but will be inverted.

5. Joining one cable to another.



If one cable with characteristic impedance Z_1 is connected to another long cable with characteristic impedance Z_2 , which is so long that signals have not had time to reflect from the far right, then let's call the 2 functions on the left as $f(x-ut)$ and $g(x+ut)$. Since there is no reflection on the right, the backward wave is absent. Let's call the forward wave on the right as $F(x-ut)$ or F .

The equations for the left and right parts are

$$\begin{aligned} V &= f + g & V &= F \\ I &= \frac{f}{Z_1} - \frac{g}{Z_1} & I &= \frac{F}{Z_2} \end{aligned}$$

We must have continuity of voltage at the boundary

$$f + g = F \text{ when } x=0$$

Divide this trivial equation by Z_1

$$\frac{f}{Z_1} + \frac{g}{Z_1} = \frac{F}{Z_1}$$

For continuity of current the currents on the two sides must be equal

$$\frac{f}{Z_1} - \frac{g}{Z_1} = \frac{F}{Z_2}$$

Add the last 2 equations to eliminate g ;

$$2\frac{f}{Z_1} = F\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) = F\frac{Z_2+Z_1}{Z_1Z_2}$$

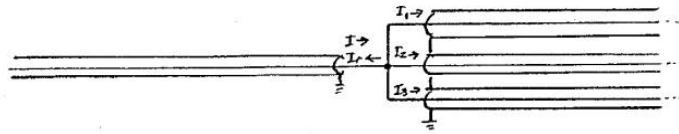
$$F = f\frac{2Z_2}{Z_2+Z_1}$$

The factor $\mathcal{T} = \frac{2Z_2}{Z_2+Z_1}$ is called the Voltage Transmission Coefficient

A challenge:

Prove that the reflected wave g here has the same formula as for the case of a simple resistor discussed before but with the notation changed slightly to have the cable with characteristic impedance Z_1 being terminated with a resistor with resistance $R = Z_2$.

6. Joining One Cable to three other cables.



Consider one Coaxial cable joining 3 other long coaxial cables with all four cables having the same Characteristic Impedance Z . Using the subscripts f and g for the original and reflected voltages & currents and the subscripts 1, 2 and 3 for the voltage & current in the three other cables, the equations to be solved are:

$$I_f - I_g = I_1 + I_2 + I_3$$

$$V_f + V_g = V_1 = V_2 = V_3$$

$$V_f = +ZI_f, \quad V_g = -ZI_g,$$

$$V_1 = V_2 = V_3 = +ZI_1 = +ZI_2 = +ZI_3$$

All “3 other long cables” will receive a transmitted signal which is \mathcal{T} times the original signal $V_f = f(x - ut)$.

$$\mathcal{T} = \frac{2\frac{Z}{3}}{(\frac{Z}{3} + Z)} = \frac{2Z}{Z + 3Z} = +\frac{1}{2}$$

The “One Coaxial cable” will receive an inverted reflection which is \mathcal{R} times the original signal.

$$\mathcal{R} = \frac{\frac{Z}{3} - Z}{\frac{Z}{3} + Z} = \frac{1 - 3}{1 + 3} = -\frac{1}{2}$$

7. When does a Cable act like a Resistor?

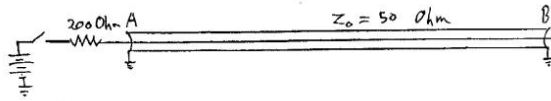
We have seen that a transmission line acts exactly like a resistor to any circuit driving it until a reflection (the backward wave g) can return. As a typical speed of signals in cables is about 0.5 of the speed of light – a typical speed in cable of 1.5×10^8 m/s or 150 mm/ns, the signal can propagate though a 1.5 meter cable in about 10 ns and its reflection can return 20 ns after the signal was first injected. Thus such a transmission line acts as a resistor, to any external system driving it, for about 20 ns.

How long must a cable be to act like a resistor for 1 μ s?

If a transmission line acts like a resistor to any external system driving it, WHY DOES THE TRANSMISSION LINE NOT GET HOT?

8. Consider charging up a 50 ohm Transmission line via a 200 ohm resistor from a battery with voltage E .

Consider the action when the switch from the battery is closed (ie connected). Assume that the signals can propagate from one end to the other in a time T .



- (a) During the first $2T$, the transmission line acts as a simple 50 ohm resistor to the battery, switch and 200 ohm resistor. Thus, the junction point A rises as a “step function” to a voltage

$$V_A = \frac{50}{200+50} \cdot E = 0.2 \times E$$

- (b) During the first T , this signal (a $0.2E$ step function) propagates down the cable and meets the open far end B. It reflects at the far end with a voltage reflection coefficient of $\mathcal{R} = +1.0$ and so a reflected $0.2E$ step function comes back. Thus after time T , the voltage at the far end rises to $V_B = \text{forward wave} + \text{reflected wave} = 2 \times 0.2 \times E = 0.4E$

- (c) At the end of the first $2T$, the reflected step function ($0.2E$) meets the near end A. Seen from the cable, the resistance to ground for any voltage change is (200 ohm + any resistance of the battery). For all normal batteries, the internal resistance is very small and much smaller than 200 ohm so we can neglect it. Thus the signal meets 200 ohm to ground.

Part of the signal ($0.2E$) reflected signal is reflected again with a voltage reflection coefficient of

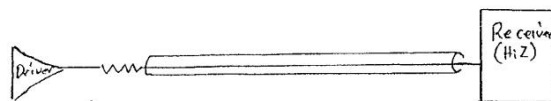
$$\mathcal{R} = \frac{200-50}{200+50} = \frac{150}{250} = 0.6$$

and so a reflection of a reflection, now $0.6 \times 0.2 \times E = 0.12 \times E$, will be sent to towards the far end again.

- (d) At time $3T$, the reflection of a reflection, $0.12E$, will reach B and bounce there again with $\mathcal{R} = +1.0$ adding $2 \times 0.12E = 0.24E$ to the $0.4E$ already there and reaching $0.64E$.
- (e) At time $4T$, the third reflection will reach A again and bounce with $\mathcal{R} = +0.6$ and so on.

The result is a series of diminishing steps at both ends of the cable gradually rising up until both ends are nearly at voltage E of the battery.

9. “Source Termination or Series Termination with $R = Z_0$ ”



The first example with a resistor $R = Z_0$ is sometimes called using parallel termination. While it has the advantage of causing reflections, it has the disadvantage of requiring a steady current from the driving circuit to maintain any unchanging signal. When it is important to minimize the power spent by the driver, some folk use series resistors as shown above.

Note that we assume here that the driver has a low output impedance and the receiving module has a high input impedance.

[Be careful, the terms parallel termination and series termination are often mis-used. For example, one can buy “series terminations for SCSI busses on computers. Although these little modules are plugged in series to the other modules on a SCSI cable, the SCSI cable carries its own grounds and supplies and the module is really a parallel termination connecting each signal line via a suitable resistor to a voltage of about +3V.]

The signal of voltage E injected on the left passes through a resistor with resistance $R = Z_0$ before it reaches the transmission line. Thus for the first few nanoseconds, the signal on the cable is $0.5E$.

This signal travels to the far end where it bounces with voltage reflection coefficient $\mathcal{R} = +1.0$ and the signal received by the receiving module is $2 \times 0.5E = E$.

The reflection travels back to the near end where it “sees” a total resistance to ground of $R = Z_0$. Here the voltage reflection coefficient $\mathcal{R} = 0$ and no further reflections occur.

Advantage of Source Termination

– lower power load in the resistors and the drivers since there is no current for steady DC signals.

Disadvantage of Source Termination

– if the transmission line has two or more receiving units, then the unit near the end of the transmission line sees a single clean transition from 0 to E . However, a unit part-way along the transmission line will see a voltage step from 0 to $0.5E$ then, later, a second voltage step from $0.5E$ to E .

The extra step will change the final signal shape and may confuse a number of analog signal recorders. Many digital circuits will misbehave if given an input which is half-way between logical 0 and logical 1.

10. Consider a sinusoidal wave form for both the forward wave and the backward wave with equal amplitudes and parameters.

$V = \textit{forward} + \textit{backward}$

$$V = a \sin\left[\frac{2\pi x}{\lambda} - \omega t\right] + a \sin\left[\frac{2\pi x}{\lambda} + \omega t\right]$$

This can be changed using $\text{Sin}A + \text{Sin} B = 2 \text{Sin}((A+B)/2)\text{Cos}((A-B)/2)$ to

$$V = a 2 \sin\left[\frac{2\pi x}{\lambda}\right] \cos[\omega t]$$

Note that the dependence upon position x and t has been separated and even though two signals are involved there appears to be no propagation. Also the signal at any point is simply an alternative voltage.

Signals like these can be caused by simply reflecting a sinusoidal input from an open or very high impedance at the far end, appear as standing waves. At some positions, the amplitude of the alternating voltage is zero and at other positions the amplitude of the alternating voltage can be $2a$.

11. A puzzle

Regard the jumper cable you use to start a car with a flat battery from another car as a perfect transmission line with characteristic impedance $Z_0 = 100 \text{ ohm}$ with both batteries having an internal resistance of about $\frac{1}{10} \text{ ohm}$. How does the voltage at the flat battery vary when you connect the jumpers and how is energy transferred to the flat battery?

9 Capacitive Termination

Consider a transmission line with a purely capacitive termination. The transmission line has an impedance Z_0 and the termination has a capacitance C_T to ground.

What will be the effect due to a step function wave in the cable meeting this termination?

Define the distance x along the cable as being $x = 0$ at the termination and x being negative in the actual cable. Assume that the step function reaches the termination $x = 0$ at time $t = 0$. We use the general solution functions we found before, $f(x - ut)$ to describe a wave travelling towards positive x and $g(x + ut)$ to describe a wave travelling towards negative x .

The voltage functions can be

$$V = f(x - ut) + g(x + ut)$$

and the corresponding current function is then

$$I = \frac{f(x-ut)}{Z_0} - \frac{g(x+ut)}{Z_0}$$

Of the two functions, we know the step function $f(x - ut)$ but we DO NOT YET KNOW the reflected function $g(x + ut)$. For the step function, the function $g(x - ut)$ is, say, $V(x - ut) = 0$ when $x - ut > 0$ and $V(x - ut) = G$ when $x - ut < 0$.

In other words, the step signal coming from far away in the cable has the value $V = 0$ then suddenly rises in a step to the value $V = G$ when $x - ut = 0$.

We label everything at the Termination with the subscript "T". At the capacitor with

capacitance C_T , the current combination of the step $f(x - ut)$ travelling towards +x and its unknown reflection $g(x+ut)$ travelling towards -x will provide a current to charge the capacitor. If V_C and I_C are the voltage and charging current at the capacitor, then the two equations for the voltage and current of the two travelling waves are:

$$V_T = f + g$$

$$I_T = \frac{f-g}{z_0}$$

On the capacitor C_T , the charge is $Q_T = C_T V_T$ and the voltage rises due to the current

$$\frac{dV_T}{dt} = \frac{1}{Z_0 C_T} (f - g)$$

$$\frac{dV_T}{dt} = \frac{1}{Z_0 C_T} (f - (V_T - f))$$

$$\frac{dV_T}{dt} = \frac{1}{Z_0 C_T} (2f - V_T)$$

After $t = 0$ at $x = 0$, $f = G$ and $V = V_T$ and so

$$\frac{dV_T}{dt} = -\frac{1}{Z_0 C_T} (V_T - 2G) \text{ "differential equation"}$$

This has a solution for the voltage V_T at the termination:

$$V_T - 2G = K e^{\frac{-t}{Z_0 C_T}}$$

[Proof. Differentiate the tentative solution and get: $\frac{V_T}{dt} = -\frac{1}{Z_0 C_T} K e^{\frac{-t}{Z_0 C_T}}$. Then substitute the differential equation to eliminate $K e^{\frac{-t}{Z_0 C_T}}$. This gives $\frac{V_T}{dt} = -\frac{1}{Z_0 C_T} (V_T - 2G)$ which satisfies the differential equation.]

Find the value of K . At $t = 0$,

$$V_T - 2G e^0 = K$$

$$0 - 2G = K$$

1. The general solution after $t = 0$ is

$$V_T - 2G = -2G e^{\frac{-t}{Z_0 C_T}}$$

$$V_T = (1 - e^{\frac{-t}{Z_0 C_T}}) 2G$$

Since $V = f + g$ and after $t = 0$, we still have $f = G$

$$g = V_T - G$$

$$g = (1 - 2e^{\frac{-t}{Z_0 C_T}}) G$$

and in the general case with $x < 0$;

$$g(x + ut) = (1 - 2e^{\frac{-(x+ut)}{Z_0 C_T}}) G$$

Using the signal speed $u = \sqrt{\frac{1}{LC}}$ and cable impedance $Z_0 = \sqrt{\frac{L}{C}}$,
 where L is the inductance per unit length,
 C is the capacitance per unit length,
 C_T is the terminating capacitance and
 G is the step of the wave travelling towards the termination,
 we have $uZ_0 = L$ and so

$$g(x + ut) = (1 - 2e^{-\frac{(x+ut)}{LC_T}})G$$

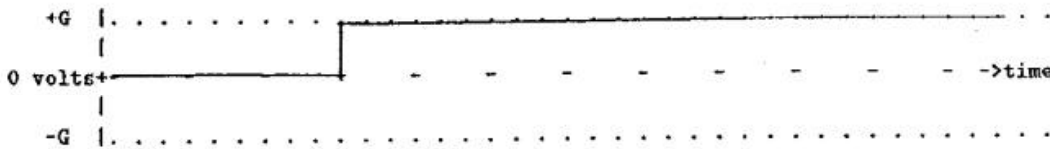
2. The general solution for $g(x + ut)$ before $t = 0$ is $g = 0$

From the two items above, the reflected waveform is $g(x + ut)$ where;

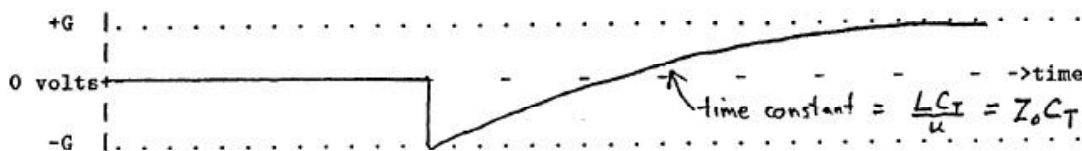
1. if $x + ut > 0$ then $g(x + ut) = (1 - 2e^{-\frac{-t}{Z_0 C_T}})G$
2. if $x + ut < 0$ then $g(x + ut) = 0$

The reflected wave due to a positive step G , starts with a negative step to $-G$ and is followed by an exponential rise to $+G$. An oscilloscope at any point will see an addition of the initial step wave $f(x - ut)$ and reflected wave $g(x + ut)$.

The initial step wave $f(x - ut)$ at an arbitrary x will show the trace on the scope:



The reflected wave $g(x + ut) = (1 - 2e^{-\frac{(x+ut)}{LC_T}})G$ at an arbitrary x will show the trace on the scope:



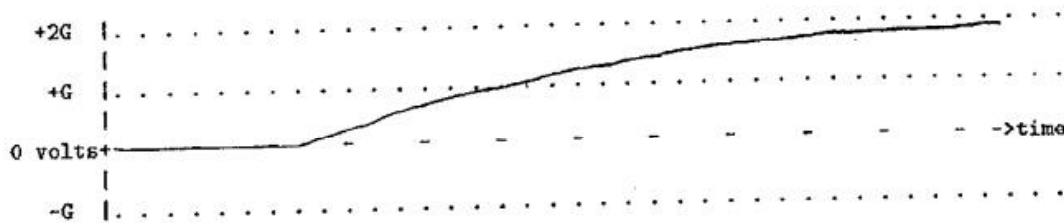
Some folk remember the shape of this reflected wave $g(x + ut)$ by saying that

1. the leading edge of the step is formed from the high frequencies being in phase and these “see” the capacitor as a short circuit giving a leading negative reflection.
2. the trailing level of the step is formed from the low frequencies being in phase and these “see” the capacitor as an open circuit giving a trailing positive reflection.

The following examples show how the forward and backward wavefronts may be seen.

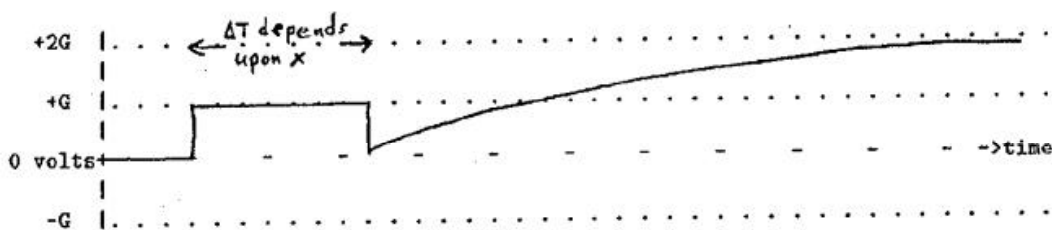
- If the scope is used to look at the voltage at the termination (at $x = 0$ due to a step, then the scope will show the sum of the two previous traces but the two sharp transients at $x = 0, t = 0$ will cancel giving:

Combined signal $f(x - ut) + g(x + ut)$ on scope:



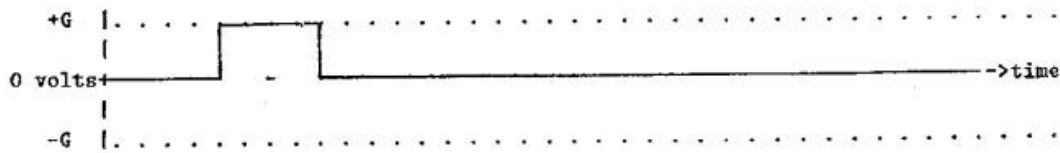
- If the scope is used to look at the signal further back on the cable (at a negative x) due to a step, then the two sharp transients will appear at different times and will NOT cancel. The scope may show:

Combined signal $f(x - ut) + g(x + ut)$ on scope:



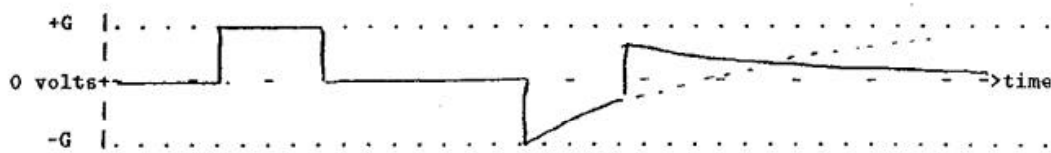
- Consider a short pulse $f(x - ut)$ instead of a step. At a long distance before the termination, the input pulse may appear, long before any reflection can arrive, as:

Short Pulse signal $f(x - ut)$ on scope at large negative x :

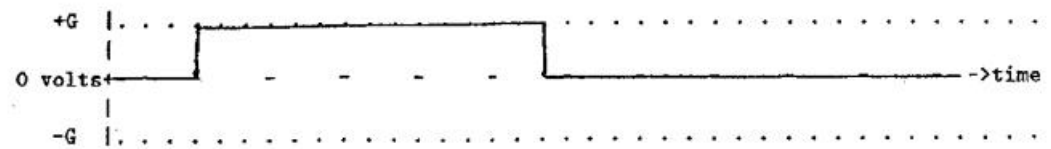


Then if the scope is used to look at the signal closer to the termination on the cable (still at a negative x) **due to $f(x - ut)$ being a positive going pulse**, then the scope may show.

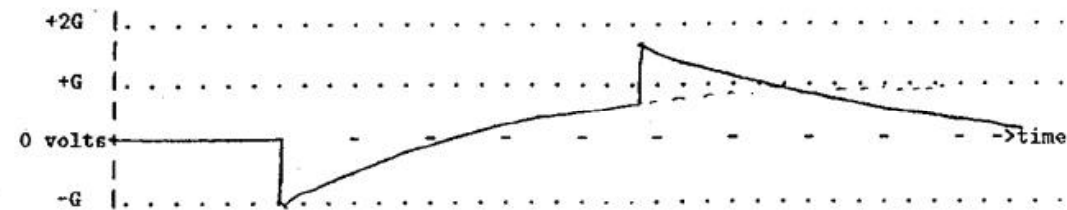
Combined signal $f(x - ut) + g(x + ut)$ on scope:



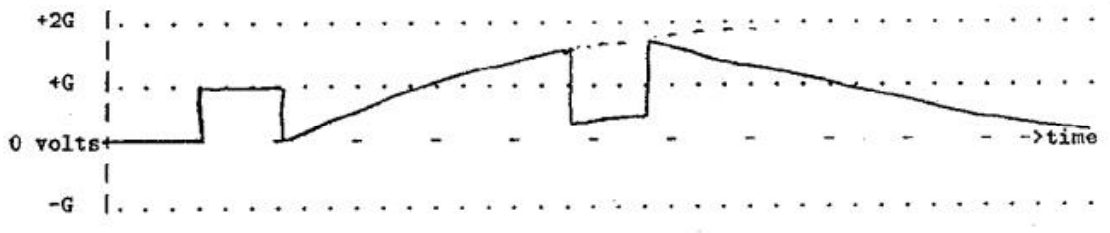
- If the input pulse is lengthened, so that long before any reflections occur, the trace looks like:



The reflection $g(x + ut)$ will look like:



then the scope will show the Combined signal $f(x - ut) + g(x + ut)$;



10 Types of Transmission Lines

(Brief Notes Follow)

Any pair of parallel conductors which have a cross and shape which are constant (independent of distance) and are far from other conductors

or any pair of parallel conductors which have a cross and shape which do not change rapidly and retain the same ratios of dimensions (independent of distance along the pair) and are far from other conductors.

If one conductor does not fully enclose the other conductor (eg twisted pair), then some of the EM field will gradually radiate away and the signal will show a steady attenuation and exponential decay which is frequency dependent.

Examples:

1. An example could be a single cylindrical conductor (wire) above a conducting plane with the wire diameter changing slowly and the distance between the wire center and the conducting plane being kept proportional to the wire diameter.
2. Twisted pair wire – often with $Z_0 \approx 120 \text{ ohm}$
3. 60 Hertz High Voltage Power Lines across the country. Strictly speaking these are usually a combination of 6 interacting transmission lines with the phase at 60 degree intervals.

The speed of power transmission is very close to that of light $u_{air} = \frac{1}{\sqrt{\mu_{air}\epsilon_{air}}} \approx \text{Velocity of light}$.

4. Ethernet coaxial cables all around Sterling Hall and Chamberlin Hall. Both have $Z_0 = 50 \pm 1 \text{ ohm}$. There are two types; yellow “thick ethernet” cables have large conductors to have minimum resistance attenuation, black RG58/U cables “thin ethernet” are thinner for least cost.
5. Electrical Transmission Lines are not Waveguides or Light Guides.
6. The 60 Hertz power lines across the country are imperfect. There is some leakage or radiation of the 60 Hertz field since the conductors are open. There is a financial incentive to arrange the phases of the 6 conductors so that the minimum radiation occurs. There is other attenuation due to slight breakdown on the insulators and corona in the air. On the average about the USA, there is about a 10% power loss??)
7. Besides the loss of power by radiation, there is another problem of open conductors — unwanted coupling to other circuits. This is not a problem at the lower frequencies but can be awful at high frequencies.

11 Imperfections of Transmission Lines

1. Radiation from open conductors mentioned above

Use Coaxial transmission lines if these are practical.

2. Skin Effect losses

These are due to the actual currents being confined by the skin effect to the surface layers of the conductors. (EM Theory)

Most current occurs within a distance “skin thickness” δ where

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \text{ where } \omega = 2\pi \times \text{frequency.}$$

Consider 3 examples

- For copper, which is pure and has been annealed, at 20 C, conductivity $\sigma_{Cu} = \frac{1}{\text{Resistivity}} = 5.80 \times 10^7 \text{ mho/m}$

$$\mu = \mu_{air} = \frac{4\pi}{10^7} \text{ H/m}$$

$$\omega = 2\pi \times \text{frequency giving}$$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{0.0661 \text{ meter.hertz}^{\frac{1}{2}}}{\sqrt{\text{Frequency}}}$$

Starting at the surface of the conductor, the current density J drops to $\frac{1}{e}$ at depth δ and a total of 1% of the total current exists at depth beyond 5δ .

For example, in thick copper lines carrying power at 60 Hertz,

$$\delta_{60 \text{ Hertz}, Cu} = \frac{0.0661}{\sqrt{60}} = 0.0085 \text{ meter} = 8.5 \text{ mm!!!}$$

Thus at 60 Hertz, only the outer centimeter of the 8 or 10 cm diameter lines carry most of the current.

- The thick aluminum power lines used for high voltage distribution have a conductivity of $\sigma_{Al} = 3.77 \times 10^7 \text{ mho/meter}$.

$$\delta_{60 \text{ Hertz}, Al} = \sqrt{\frac{2}{\omega\mu\sigma_{Al}}} = 0.0105 \text{ meter} = 10.5 \text{ mm!!}$$

[Why is aluminum, rather than copper, used for our National power system? Calculate the resistance of a 10,000 km line or rod for both materials assuming that the effective cross sectional area used in each line is $2\pi r\delta$ and the radius is $r = 100 \text{ mm}$. Also calculate the cost of each line or rod using; Aluminum conductivity $\sigma_{Al} = 3.77 \times 10^7 \text{ mho/m}$, density = 2708 kg/m^3 cost/kg = 7.70 \$/kg.

Copper conductivity $\sigma_{Cu} = 5.80 \times 10^7 \text{ mho/m}$, density = 2567 kg/m^3 cost/kg = 15.40 \$/kg.]

Do the centers of the heavy power lines have much use?

- For copper lines at frequencies near 60 MHz, $\delta = \frac{0.0661 \text{ meter.hertz}^{\frac{1}{2}}}{\sqrt{60 \times 10^6 \text{ hertz}}} = 8.5 \text{ microns!!!}$
Since the skin depth is frequency dependent, the effect is to attenuate the high frequencies of the signal more rapidly than the low frequencies of the signal. Thus the signal CHANGES ITS SHAPE during the propagation due to the skin effect.

11.1 “Remedies” for Signal Loss

On copper at high frequencies, the skin depth (8.5 microns at 60 MHz) can be serious.

1. The skin depth of $\delta = 8.5$ microns may have surface cracks, surface oxide, and other defects which increase its resistance and further attenuate the signal. A good conductor has higher σ and smaller δ and although the depth $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$ varies only as $\sqrt{\frac{1}{\sigma}}$, it is still best to use high σ to reduce the signal loss. Therefore at high frequencies, use gold or silver plating. Why is gold used upon some critical circuits when copper costs less and copper is the better conductor? A few conductivities are listed in decreasing order.

| Metal | Metal Conductivity | |
|---------------|--|--|
| Silver | $\sigma_{Ag} = 6.15 \times 10^7$ mho/meter | The best conductor. (Avoid AgS) |
| Copper | $\sigma_{Cu} = 5.80 \times 10^7$ mho/meter | This is used on central Al conductor of Al cables TV cables. (Avoid CuO) |
| Gold | $\sigma_{Au} = 4.10 \times 10^7$ mho/meter | This does not tarnish! |
| Aluminum rods | $\sigma_{Al} = 3.77 \times 10^7$ mho/meter | Used for most high voltage power lines. |

2. Use smooth polished solid conductors to minimize the distances for the currents near the surfaces. (Unfortunately, to gain a flexible cable, often the central conductor is multistranded and the outer conductor is made of tinned braided thin copper.
3. Use big conductors (large r) – the conducting cross sectional area A is about $A \approx 2\pi r \times \delta$ although the resistance and power loss at a given current I , frequency $f = \frac{\omega}{2\pi}$ and conductivity σ is only proportional to $\frac{1}{r}$.

The cost of the central conductor of a coaxial cable (transmission line), of course, is proportional to Lr^2 .

4. The cross sectional conducting area is about $A = 2\pi r \delta = 2\pi r \sqrt{\frac{2}{\omega\mu\sigma}}$. The resistance per unit length R is about $R = \frac{1}{\sigma A} \approx \frac{1}{\sigma 2\pi r \delta} \approx \frac{1}{\sigma 2\pi r} \sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1}{2\pi r} \sqrt{\frac{\omega\mu}{2\sigma}}$

To reduce the power loss per unit length, $I^2 R \approx \frac{I^2}{2\pi r} \sqrt{\frac{\omega\mu}{2\sigma}}$, use good conductors (high σ) although σ is in a square root and cannot help much.

5. For high powered long distance systems, use cheap metals.
6. For high powered systems to deliver a particular power level $P = IV$, increase to a really high voltage V to reduce I and reduce the power loss.
7. For high powered long distance systems, use metals with a high tensile strength to span between the pylons and so need fewer pylons. This leads to a compromise between

conductivity σ and tensile strength. If practical, in each wire or rod carrying current, use a high tensile inner wire surrounded by high conductivity but weaker outer wires. Then worry about corrosion!

8. Beware of damaged coaxial cable causing unwanted reflections and reductions in the forward signal. Crushed coax has a lower Z_0 and the lumpy dielectric can change Z_0 – both cause reflections.

9. Speeds in some cables can be SLOW! Even if no magnetic materials are used (ie $\mu = \mu_0$), then

$$u = \frac{c}{\sqrt{\epsilon_r}}$$

If necessary, use a foam dielectric with $\epsilon_r \approx 1$ giving u close to the speed of light.

10. Minimize the resistance of the outer conductor.

11. Minimize electrical breakdown.

12. Choose your connectors carefully. Even the coaxial connectors often have a Z_0 which is different from that of the cables and cause a pair of reflections which have the same shape and opposite sign but do not cancel because they occur at slightly different times.

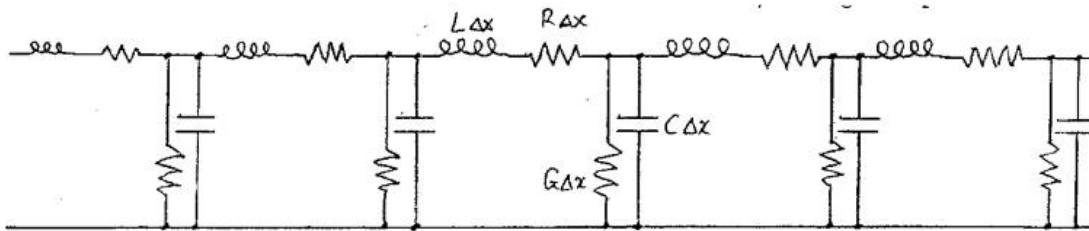
Noncoaxial connectors often have short links with relatively high characteristic impedance and these impedance transitions can cause a number of upright and inverted reflections which are close together but not cancelling.

13. For small signals being sent over long distance systems, use optical fibers!

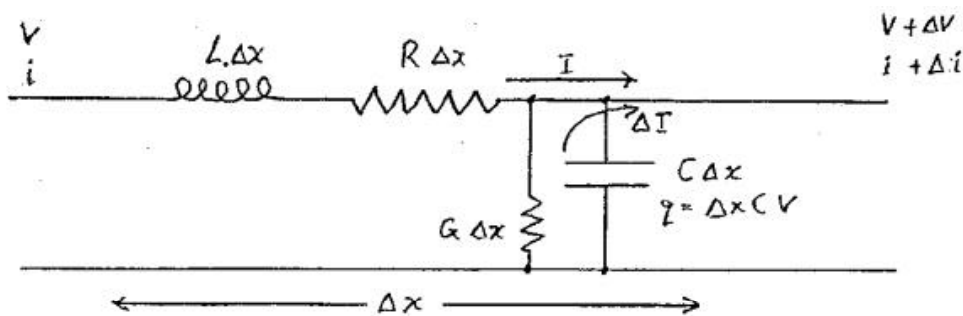
12 Equations for a “Lossy” Transmission Line

In section 2 before, on page 2, we began to consider “Lossless” transmission lines. However, sometimes, we must use transmission lines with the series resistance/unit length being non-zero and the insulation being non-perfect with a parallel conductance of G being non-zero. A useful simple reference is “Theory and Problems of Transmission Lines” by Robert A. Chipman (Schaum Outline Series 1968). We will continue to use x for the distance along the transmission line whereas Chipman uses z for this. Our x will avoid confusion between the distance and the characteristic impedance Z .

Consider a transmission line with distributed resistance R ohm/unit length and parallel conductance G mho/unit length.



As before, consider one such cell corresponding to the components between position x and position $x + \Delta x$ along the transmission line.



The resistance per unit length, R , includes both the resistance of the inner conductor and that of the outer conductor. Often, to minimize the total weight, minimize the cost or maximize the flexibility, the outer conductor is made thin with braided thin wires and can contribute significantly to the total R . In the theory of propagation, we are only concerned with the total resistance per unit length (inner + outer resistance/unit length) and call this total as " R ".

The equations for each element Δx of length are

$$v(x + \Delta x, t) - v(x, t) = -R\Delta x \cdot i(x, t) - L\Delta x \cdot \frac{\partial i(x, t)}{\partial t}$$

$$i(x + \Delta x, t) - i(x, t) = -G\Delta x \cdot v(x, t) - C\Delta x \cdot \frac{\partial v(x, t)}{\partial t}$$

Dividing by Δx and letting $\Delta x \rightarrow 0$, we get similar equations to equations 3.3 and 3.4 of Chipman.

$$\frac{\partial v(x, t)}{\partial x} = -Ri(x, t) - L\frac{\partial i(x, t)}{\partial t}$$

$$\frac{\partial i(x, t)}{\partial x} = -Gv(x, t) - C\frac{\partial v(x, t)}{\partial t}$$

This can be written in the most simple form, remembering that v and i are functions of x and t ;

$$\frac{\partial v}{\partial x} = -Ri - L\frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial x} = -Gv - C \frac{\partial v}{\partial t}$$

These equations are similar, of course, to the equations 2 and 1 which we obtained upon page 3 but now include the non-zero R and G .

As before, we can substitute for i in one equation from the other to obtain a differential equation in v . Similarly we can obtain a differential equation for i .

$$\frac{\partial^2 v}{\partial x^2} = LC \cdot \frac{\partial^2 v}{\partial t^2} + (LG + RC) \cdot \frac{\partial v}{\partial t} + RGv$$

$$\frac{\partial^2 i}{\partial x^2} = LC \cdot \frac{\partial^2 i}{\partial t^2} + (LG + RC) \cdot \frac{\partial i}{\partial t} + RGi$$

Notes

1. These equations do not have the simple form and simple solution of the equations for a “lossless” line.
2. Although we have assumed that the L , C , R and G are constants, at high frequencies, these can be frequency dependent. For example, they may be influenced by the skin effect which is frequency dependent.

Never the less, if we consider just one Fourier component of a signal, we can obtain an understanding, then by combining the Fourier components of any particular signal, we can understand how a particular signal propagates.

3. Although the equations above for v and i are identical, they will usually have different boundary conditions and so will have different solutions.

We can replace $v(x, t)$ by $Re\{V(x)e^{j\omega t}\}$ and $i(x, t)$ by $Re\{I(x)e^{j\omega t}\}$

$$\frac{dV(x)}{dx} = -(R + j\omega L)I(x)$$

$$\frac{dI(x)}{dx} = -(G + j\omega C)V(x)$$

From these we get two second order differential equations similar to Equations 3.15 and 3.16 of Whitman.

$$\frac{d^2 V}{dx^2} - (R + j\omega L)(G + j\omega C)V = 0$$

$$\frac{d^2 I}{dx^2} - (R + j\omega L)(G + j\omega C)I = 0$$

We define γ where $\gamma^2 = (R + j\omega L)(G + j\omega C)$.

$$\frac{d^2 V}{dx^2} - \gamma^2 V = 0$$

$$\frac{d^2 I}{dx^2} - \gamma^2 I = 0$$

The solutions for these are voltages and currents with an angular frequency ω .

$$V(x) = V_1 e^{-\gamma x} + V_2 e^{+\gamma x}$$

$$I(x) = I_1 e^{-\gamma x} + I_2 e^{+\gamma x}$$

where V_1, V_2, I_1 and I_2 are arbitrary constants and where $\gamma^2 = (R + j\omega L)(G + j\omega C)$. Define α and β as the real and imaginary parts of γ so that

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Then the solution for V is

$$V(x, t) = [e^{j\theta_1} \cdot V_1 e^{-(\alpha x + j\beta x)} + e^{j\theta_2} \cdot V_2 e^{+(\alpha x + j\beta x)}]$$

$$V(x, t) = [e^{j\theta_1} \cdot V_1 e^{-\alpha x} e^{-j\beta x} + e^{j\theta_2} \cdot V_2 e^{+\alpha x} e^{+j\beta x}]$$

Using $v(x, t) = \text{Re}\{V \cdot e^{j\omega t}\}$, this becomes

$$v(x, t) = \text{Re}\{e^{j\omega t} \cdot [e^{j\theta_1} \cdot V_1 e^{-\alpha x} e^{-j\beta x} + e^{j\theta_2} \cdot V_2 e^{+\alpha x} e^{+j\beta x}]\}$$

$$v(x, t) = V_1 e^{-\alpha x} \text{Re}\{e^{j(\omega t - \beta x + \theta_1)}\} + V_2 e^{+\alpha x} \text{Re}\{e^{j(\omega t + \beta x + \theta_2)}\}$$

corresponding to a forward (x increasing) wave + a backward (x decreasing) wave. The parts of this equation can be identified.

- V_1 is an arbitrary amplitude for a signal propagating in the +x direction.
- $e^{-\alpha x}$ is a real coefficient diminishing in the forward or +x direction and describing the attenuation of the signal as it moves in that direction.
- $(\omega t - \beta x)$ is a term describing a wave propagating in the + x direction with phase speed $u = +\frac{\omega}{\beta}$ since if t is increased by Δt , then the value is unchanged if the x is increased by $\Delta x = +\frac{\omega}{\beta} \Delta t$. The group speed is $= +\frac{d\omega}{d\beta}$.
- θ_1 is an arbitrary phase angle
- $e^{j(\omega t - \beta x + \theta_1)}$ is a sinusoidal wave of angular frequency ω propagating in the + x direction with phase speed $= +\frac{\omega}{\beta}$ and group speed $= +\frac{d\omega}{d\beta}$.
- V_2 is an arbitrary amplitude for a signal propagating in the -x direction.
- $e^{+\alpha x}$ is a real coefficient diminishing in the backward or -x direction and describing the attenuation of the signal as it moves in that direction.
- $(\omega t + \beta x)$ is a term describing a wave propagating in the - x direction with phase speed $u = -\frac{\omega}{\beta}$ since if t is increased by Δt , then the value is unchanged if the x is decreased by $\Delta x = \frac{\omega}{\beta} \Delta t$. The group speed is $= -\frac{d\omega}{d\beta}$.
- θ_2 is an arbitrary phase angle
- $e^{j(\omega t + \beta x + \theta_2)}$ is a sinusoidal wave of angular frequency ω propagating in the - x direction with phase speed $u = -\frac{\omega}{\beta}$ and group speed $= -\frac{d\omega}{d\beta}$.

- $v(x,t)$ is the instantaneous voltage at position x and time t .

13 Lossy Line with No Reflections

If only the forward wave exists, the equations become;

$$V = V_1 e^{-\alpha z} e^{-j\beta z}$$

$$I = I_1 e^{-\alpha z} e^{-j\beta z}$$

14 Attenuation if line is slightly lossy

α and β may be written

$$\alpha + j\beta = \sqrt{(R + j\omega L) \cdot (G + j\omega C)}$$

$$\alpha + j\beta = j\omega\sqrt{LC} \cdot \left[\left(\frac{R}{j\omega L} + 1 \right)^{\frac{1}{2}} \cdot \left(\frac{G}{j\omega C} + 1 \right)^{\frac{1}{2}} \right]$$

Now use a Taylor expansion of each $()^{\frac{1}{2}}$ as power series of $\frac{R}{j\omega L}$ and $\frac{G}{j\omega C}$

$$\alpha + j\beta = j\omega\sqrt{LC} \cdot \left[\left(\dots + \left(\frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \times 2} \right) \cdot \left(\frac{R}{j\omega L} \right)^2 + \frac{1}{2} \cdot \frac{R}{j\omega L} + 1 \right) \cdot \left(\dots + \left(\frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \times 2} \right) \cdot \left(\frac{G}{j\omega C} \right)^2 + \frac{1}{2} \cdot \frac{G}{j\omega C} + 1 \right) \right]$$

$$\alpha + j\beta = j\omega\sqrt{LC} \cdot \left[\left(\dots - \frac{1}{8} \cdot \left(\frac{R}{j\omega L} \right)^2 + \frac{1}{2} \cdot \frac{R}{j\omega L} + 1 \right) \cdot \left(\dots + \frac{1}{8} \cdot \left(\frac{G}{j\omega C} \right)^2 + \frac{1}{2} \cdot \frac{G}{j\omega C} + 1 \right) \right]$$

L and C are usually log functions of ratios of cable radii or other cable dimensions and so cannot be very high or very small. However, if R and G are small but non-zero OR the frequency $f = \frac{\omega}{2\pi}$ is very high, so that $\frac{R}{j\omega L} \ll 1$ and $\frac{G}{j\omega C} \ll 1$, then these two Taylor series can be simplified by neglecting the higher powers of $\frac{R}{j\omega L}$ and $\frac{G}{j\omega C}$.

$$\alpha + j\beta = j\omega\sqrt{LC} \cdot \left[\left(\frac{R}{j2\omega L} + 1 \right) \cdot \left(\frac{G}{j2\omega C} + 1 \right) \right]$$

Equating the real parts;

$$\alpha = \omega\sqrt{LC} \cdot \left[\frac{R}{2\omega L} + \frac{G}{2\omega C} \right]$$

$$\alpha = \frac{\sqrt{LC}}{2} \cdot \left[\frac{R}{L} + \frac{G}{C} \right]$$

Equating the imaginary parts;

$$j\beta = j\omega\sqrt{LC} \cdot \left[1 + \left(\frac{R}{2j\omega L} \frac{G}{2j\omega C} \right) \right]$$

$$\beta = \omega\sqrt{LC} \cdot \left[1 - \frac{RG}{4\omega^2 LC} \right]$$

$$\beta \approx \omega\sqrt{LC} \quad (\text{neglecting the product of } \frac{R}{j\omega L} \text{ and } \frac{G}{j\omega C}).$$

From β , we can obtain the phase speed $u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$, and group speed $= \frac{d\omega}{d\beta} = \frac{1}{\sqrt{LC}}$.

[Make a check of these equations for the lossless transmission line. If $R = G = 0$, then

$\alpha = \frac{\sqrt{LC}}{2} \cdot [0 + 0] = 0$ and so, as expected, there is no attenuation of either the forward wave or backward wave.

$$\beta = \omega\sqrt{LC} \cdot [1 - 0] = \omega\sqrt{LC}$$

The full equation can be written for the lossless case.

$$v(x, t) = V_1 e^{-\alpha x} \operatorname{Re}\{e^{j(\omega t - \beta x + \theta_1)}\} + V_2 e^{+\alpha x} \operatorname{Re}\{e^{j(\omega t + \beta x + \theta_2)}\}$$

becomes

$$v(x, t) = V_1 \operatorname{Re}\{e^{j(\omega t - \beta x + \theta_1)}\} + V_2 \operatorname{Re}\{e^{j(\omega t + \beta x + \theta_2)}\}$$

15 Characteristic Impedance of a Lossy Transmission Line

We get $\frac{V}{I} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$

This is the characteristic impedance and can be split into real and imaginary parts

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$Z_0 = R_0 + jX_0$$

If the transmission line is only slightly lossy, then R and G are small and, for most purposes, we ignore X_0 .

16 Heavyside Distortionless Lines

If $\frac{R}{L} = \frac{G}{C}$, then it can be shown that the speed of propagation is the same for all angular frequencies ω and the shape of the signal with respect to position X , remains constant although it gradually gets smaller with the attenuation.

Unfortunately, this idea has little practical use because the R , L , G and C are sufficiently frequency dependent for the above theory to be insufficient.

17 Three Examples

Electrical parameters from the Alpha and Belden handbooks and dimensions from the CRC Handbook of C&P pg F-117.

L is obtained from $L = Z^2 C$ and $L = \frac{Z}{speed}$.

| Name | Maker | Impedance Z | speed | C | L | R_{DC} | G | d | wire gauge | cross sectl area |
|--|---|------------------|------------|------|-----------|----------|-------|-------|---------------|------------------------|
| | | ohm | $\times c$ | pF/m | $\mu H/m$ | ohm/m | mho/m | mm | # AWG | mm^2 |
| RG58/U | Alpha 9848 pg 263 Belden 8240 pg 133 | 50.0 | 0.78 | 85.3 | 0.213 | 0.02095 | neg | 1.024 | 18 | 0.8231 |
| Thick Ethernet Yellow Non- Plenium | Alpha pg 233 | 50.0 | 0.78 | 90.2 | | 0.0046 | neg | 1.662 | | 2.17 |
| RG59 B/U | Belden 9059B | 75.0 | 0.66 | 65.9 | | | neg | | | 0.26? |

“neg” = negligible

RG58/U is a common coaxial cable used very frequently in all sciences. In the next two sections, we will consider the example of RG58/U at frequencies of 0, 10 MHz and 1000 MHz.

The specifications of a given type of cable vary from manufacturer to manufacturer. They may have slight differences (multistranded or solid central conductor, single or double outer conductor, flexible or rigid outer plastic covering, etc). Some have inconsistent values due to varying roundoff. Some have metric measures and other have inches. As a result, comparisons are difficult and the customer should be careful.

18 Attenuation at DC and Low Frequencies

Consider a coaxial cable and signal for which the frequency is sufficiently low that the skin depth is larger than the diameter d of the central wire.

- The resistance per meter at DC (ie when frequency = 0) is

$$R_{DC} = \frac{1}{\sigma \times 0.8231 \text{ mm}^2} = \frac{1}{5.80 \times 10^7 \text{ mho/m} \times 0.8231 \times 10^{-6} \text{ m}^2} = 0.209 \times 10^{-1} \text{ ohm/m} = 0.0209 \text{ ohm/m}$$

This agrees with the CRC handbook of C&P pg F-117 of $R=20.95 \text{ ohm/km}$ for annealed copper at a temperature of 20 C.

- The inductance per unit length is $L = Z^2 C = 50^2 \times 85.3 \times 10^{-12} \text{ H/m}$
 $L = 2500 \times 85.3 \times 10^{-12} \text{ H/m} = 0.213 \times 10^{-6} \text{ Henry/meter}$
- The signal speed is $0.78 \times c$ and so $u = \sqrt{\frac{1}{LC}} = 0.78 \times 3 \times 10^8 \text{ m/s}$
- The impedance Z is approximately $\sqrt{\frac{L}{C}} = 50 \text{ ohm}$
- The leakage as in most modern cables, is negligible, so set the leakage conductance $G = 0$.

We can use the speed, the impedance and $G = 0$ to obtain the attenuation. The attenuation factor α is then

$$\alpha = \frac{\sqrt{LC}}{2} \cdot \left[\frac{R}{L} + \frac{G}{C} \right]$$

$$\alpha = \frac{\sqrt{LC}}{2} \cdot \left[\frac{R}{L} \right]$$

Substitute $Z = \sqrt{\frac{L}{C}}$

$$\alpha = \frac{R}{2Z} \quad (\text{Chipman pgs 49, 55 \& 57})$$

For RG58/U with no skin effect (ie at DC or low frequencies);

$$\alpha = \frac{0.02095}{2 \times 50} / \text{meter} = 0.2095 \times 10^{-3} / \text{meter}$$

From α for any cable, we can calculate the number of db loss of that cable over a length x by

$$\text{db loss} = 10 \times \log_{10} \left(\frac{\text{Power}_{\text{input}}}{\text{Power}_{\text{at 1 km}}} \right)$$

$$\text{db loss} = 10 \times \log_{10} \left(\frac{V_{\text{input}}}{V_{\text{at 1 km}}} \right)^2$$

$$\text{db loss} = 20 \times \log_{10} \left(\frac{V_{\text{input}}}{V_{\text{at 1 km}}} \right)$$

$$\text{db loss} = 20 \times \log_{10}(e^{\alpha x})$$

$$\text{db loss} = 20 \times \frac{\log_e(e^{\alpha x})}{\log_e 10}$$

$$\text{db loss} = 20 \times \frac{\alpha x}{2.3026}$$

This is often stated simply as

$$\text{db loss} = 8.6858 \alpha x$$

The attenuation of the voltage signal, due to only the central conductor, on RG58/U over a distance of $x = 1 \text{ km} = 1000 \text{ m}$ is;

$$db\ loss = 20 \times \frac{0.2095 \times 10^{-3} / m \times 1000\ m}{2.3026} db$$

$db\ loss = 1.820\ db$ for the 1 km length at zero and low frequencies.

19 Attenuation at Higher Frequencies

19.1 Skin Effect Loss

The loss rate calculated above is for low frequencies in which all of the central conductor conducts the signal. At the higher frequencies, the skin effect reduces the cross section of the copper which conducts and so the attenuation is increased.

$$\alpha = \frac{\sqrt{LC}}{2} \cdot \left[\frac{R}{L} \right]$$

$$\alpha = \frac{1}{2\sqrt{\frac{L}{C}}} R$$

$$\alpha = \frac{R}{2Z} \quad (\text{See Chipman pgs 49 \& 85.})$$

Since for all frequencies above some lower limit, the skin effect restricts the current to approximately the skin depth of the copper

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{0.0661\ meter.hertz^{\frac{1}{2}}}{\sqrt{Frequency}}$$

we can estimate the approximate resistance/meter at these high frequencies.

For copper lines at frequencies near 10 MHz, $\delta = \frac{0.0661\ meter.hertz^{\frac{1}{2}}}{\sqrt{10 \times 10^6\ hertz}} = 20.9\ microns$. If the diameter of the central conductor is d then, for most cables and frequencies above 1 MHz, $\delta \ll d$. With this assumption then the conducting region is equivalent to that of a thin shell with a depth of δ . This relation can be proved and is called the "Skin Effect Theorem". (See Chipman pgs 78 & 85.) From this, we can estimate the effective resistance/unit length R_f , at a particular frequency f . Since the periphery is πd , the cross section of $(\pi d \times \delta)$.

$$R_f \approx \frac{1}{\sigma \times (\text{conduction cross sectional area})}$$

$$R_f \approx \frac{1}{\sigma \times (\pi d \times \delta)}$$

$$R_f \approx \frac{1}{\sigma \times (\pi d \times \sqrt{\frac{2}{\omega\mu\sigma}})}$$

$$R_f \approx \frac{1}{\pi d} \cdot \sqrt{\frac{\omega\mu}{2\sigma}}$$

Note that, at the higher frequencies, $R_f \propto \sqrt{\omega} \propto \sqrt{f}$

We estimate the attenuation due to only the central conductor of RG58/U at 10 MHz and 1.0 GHz. We do not have enough data to estimate the attenuation due to the outer conductor.

RG58/U at $f = 10$ MHz:

$$R_{10 \text{ MHz}} \approx \frac{1}{\pi \times 1.024 \times 10^{-3}} \cdot \sqrt{\frac{2\pi \times 10 \times 10^6}{2 \times 5.80 \times 10^7} \cdot \frac{4\pi}{10^7}} \text{ ohm/m}$$

$$R_{10 \text{ MHz}} \approx \frac{1}{1.024 \times 10^{-3}} \cdot \sqrt{\frac{10 \times 10^6}{5.80 \times 10^7} \cdot \frac{4}{10^7}} \text{ ohm/m}$$

$$R_{10 \text{ MHz}} \approx \frac{1}{1.024 \times 10^{-3}} \cdot \left(\sqrt{\frac{40}{5.80 \times 10^8}} \right) \text{ ohm/m}$$

$$R_{10 \text{ MHz}} \approx \frac{1}{1.024 \times 10^{-3}} \cdot (2.626 \times 10^{-4}) \text{ ohm/m}$$

$$R_{10 \text{ MHz}} \approx 2.564 \times 10^{-1} \text{ ohm/m}$$

$$R_{10 \text{ MHz}} \approx 0.2564 \text{ ohm/m}$$

From this we can estimate α for RG58/U at 10 MHz;

$$\alpha = \frac{1}{Z} \cdot \frac{1}{\pi d} \cdot \sqrt{\frac{\omega \mu}{2\sigma}}$$

$$\alpha = \frac{R}{2Z}$$

$$\alpha \approx \frac{0.2564}{2 \times 50} \approx 0.002564 \text{ /meter}$$

For 10 MHz signals in an RG58/U cable 1 km long;

$$\text{db loss} \approx 20 \times \frac{\alpha x}{2.3026} \text{ db}$$

$$\text{db loss} \approx 20 \times \frac{0.002564 \text{ /m} \times 1000 \text{ m}}{2.3026} \text{ db}$$

$$\text{db loss} \approx 22.3 \text{ db for the 1 km length at 10 MHz.}$$

RG58/U at $f = 1000$ MHz = 1.0 GHz:

$$R_{1 \text{ GHz}} \approx \frac{1}{\pi \times 1.024 \times 10^{-3}} \cdot \sqrt{\frac{2\pi \times 1.0 \times 10^9}{2 \times 5.80 \times 10^7} \cdot \frac{4\pi}{10^7}} \text{ ohm/m}$$

$$R_{1 \text{ GHz}} \approx \frac{1}{1.024 \times 10^{-3}} \cdot (2.626 \times 10^{-3}) \text{ ohm/m}$$

$$R_{1 \text{ GHz}} \approx 2.564 \times \text{ ohm/m}$$

From this we can estimate α for RG58/U at 1.0 GHz;

$$\alpha = \frac{R}{2Z}$$

$$\alpha \approx \frac{2.564}{2 \times 50} \approx 0.02564 \text{ /meter}$$

For 1.0 GHz signals in an RG58/U cable 1 km long;

$$\text{db loss} \approx 20 \times \frac{\alpha x}{2.3026} \text{ db}$$

$$\text{db loss} \approx 20 \times \frac{0.02564 \text{ /m} \times 1000 \text{ m}}{2.3026} \text{ db}$$

db loss ≈ 223 db for the 1 km length at 1.0 GHz.

19.2 Dielectric Loss

This attenuation is due to the dielectric absorbing energy as it is polarized in each direction. This loss is usually small but becomes more significant at the higher frequencies.

The effect is that our conductance/unit length becomes non-zero at the higher frequencies. The difference of the phase of the current ΔI between the central conductor and the outer conductor and the phase of the voltage changes from being $\frac{\pi}{2}$ to $\frac{\pi}{2} - \epsilon$ and data sheets can give a loss factor of “*Tan ϵ* ” to give the relation. Unfortunately, we don’t have convenient data sheets for the dielectric of RG58/U.

The usual circuit boards made from epoxy-glass have a significant attenuation due to the dielectric loss while polyethylene has a modest loss.

19.3 Radiation Loss

If the conductors form a tight electromagnetic system with the outer conductor having a thickness greater than about 5δ (5 times the skin depth) then external EM fields will be small and the radiation loss is negligible.

If the outer conductor is a loose braid, the the external EM fields will cause radiation away from the cable and will cause attenuation.

If the transmission line has two open conductors with neither shielding the other, then the external fields are minimized by twisting the two conductors (forming a “twisted pair” line). However, even if the pitch of the twist is much smaller than the predominant wavelength of the electrical signal, some radiation occurs and the signal is attenuated. For this reason, be careful when using twisted pair lines.

[In addition such twisted pair lines can easy receive electrical noise from other circuits which have high current or voltage transients.]

19.4 Actual Attenuation in Cables

We can compare the results of these approximate calculations with actual attenuation measurements.

From the Alpha handbook,

- for RG58/U (pg 273) the loss at the relatively low frequency of 10 MHz of RG58/U is 1.2 db/100 ft = 3.94 db/100 meters= 39.4 db/km.

The loss at the relatively high frequency of 1000 MHz is $18.0 \text{ db}/100 \text{ ft} = 59 \text{ db}/100 \text{ meters} = 590 \text{ db}/\text{km}$.

- for Thick Ethernet (pg 233) the loss at 10 MHz is $20 \text{ db}/\text{km}$.

Obviously, the estimate of the loss due to only the skin effect of the central conductor is about half of the db of the actual loss. Probably the remaining loss is due to the imperfect outer conductor and to the dielectric loss.(?)

- The central or inner conductor has a smaller periphery but usually has a smooth surface and high conducting metal.
- The outer conductor has a smaller periphery but is usually rough (made of braided wires for flexibility) with only tinned surfaces and imperfect contacts. The majority of the cable cost is due to the outer metal and so manufacturers may skimp a little on the conductivity of the outer conductor.