Exam 2 solns

1) a) $\sqrt{j(j+1) - m(m+1)}(m+2)\sqrt{j(j+1) - m(m+1)}\,\hbar^2 = \sqrt{3}\,2\hbar^2$

b) $[p, x^2] = px^2 - x^2 p = -i\hbar \frac{d}{dx}x^2 = -2i\hbar x$

$[x,p] = i\hbar$
$[p, f] = -i\hbar f'$

c) $W = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & \gamma & 0 \end{pmatrix}$

2) a) $|2\rangle = -\sin\theta|e\rangle + \cos\theta|\mu\rangle$

b) $\psi(0) = |\mu\rangle = \cos\theta|2\rangle + \sin\theta|1\rangle$

$\psi(t) = \cos\theta e^{-iE_2 t}|2\rangle + \sin\theta e^{-iE_1 t}|1\rangle$

$\langle e|\psi(t)\rangle = -\sin\theta\cos\theta e^{-iE_2 t} + \sin\theta\cos\theta e^{-iE_1 t}$

$|\langle e|\psi(t)\rangle|^2 = \sin^2\theta\cos^2\theta\,|e^{-iE_1 t} - e^{-iE_2 t}|^2$

3) $E'_m = E_m + \langle m|V|m\rangle$

$-3/2$	$-3/2\,\mu B + Q/8$
$-1/2$	$-1/2\,\mu B - Q/8$
$+1/2$	$1/2\,\mu B - Q/8$
$3/2$	$-3/2\,\mu B + Q/8$

4) $A^+A = \left(\frac{-ip}{\sqrt{2m}} + W\right)\left(\frac{ip}{\sqrt{2m}} + W\right)$

$= \frac{p^2}{2m} + \frac{i}{\sqrt{2m}}(\underbrace{Wp - pW}_{i\hbar W'}) + W^2$

$= \frac{p^2}{2m} + W^2 - \frac{\hbar W'}{\sqrt{2m}} = \frac{p^2}{2m} + V$

$\therefore V = W^2 - \frac{\hbar}{\sqrt{2m}} W'$

5) $H = \hbar\omega\left(a^\dagger a + 1/2 + \alpha\left(\frac{a+a^\dagger}{\sqrt{2}}\right)^4\right)$

$\langle 0|V|0\rangle = 3\alpha/4 \qquad \langle 2|V|0\rangle = \frac{3\alpha}{\sqrt{2}} \qquad \langle 4|V|0\rangle = \sqrt{\frac{3}{2}}\,\alpha$

$\Delta E = \frac{3\alpha}{4} + \frac{9\alpha^2/2}{-2} + \frac{\frac{3\alpha^2}{2}}{-4} = \frac{3\alpha}{4} - \frac{21\alpha^2}{8}$

rest on Mathematica

5a

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a†[b_. ket[n_]] := b √(n+1) ket[n+1]
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$$a[b\_.\ ket[n\_]] := b\sqrt{n}\ ket[n-1]$$

$$x[b\_.\ ket[n\_]] := \frac{a[b\ ket[n]] + a\dagger[b\ ket[n]]}{\sqrt{2}}$$

$$x[c\_.\ (a\_ + b\_)] = c\ x[a] + c\ x[b]$$

$$c\ x[a] + c\ x[b]$$

`x[x[ket[0]]] // Simplify`

$$\frac{1}{2}\left(ket[0] + \sqrt{2}\ ket[2]\right)$$

`x[x[x[x[ket[0]]]]] // Simplify // Expand`

$$\frac{3\ ket[0]}{4} + \frac{3\ ket[2]}{\sqrt{2}} + \sqrt{\frac{3}{2}}\ ket[4]$$

$$\text{Collect}\left[\frac{3\alpha}{4} + \frac{\left(\frac{3\alpha}{\sqrt{2}}\right)^2}{-2} + \frac{\left(\sqrt{\frac{3}{2}}\ \alpha\right)^2}{-4}\ //\ \text{Simplify},\ \alpha\right]$$

$$\frac{3\alpha}{4} - \frac{21\alpha^2}{8}$$

5b

`Clear[x]`

$$\left(a = \text{Table}\left[\begin{cases}\sqrt{i} & j == i-1 \\ 0 & \text{True}\end{cases},\ \{j, 0, 4\},\ \{i, 0, 4\}\right]\right)\ //\ \text{MatrixForm}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(a\dagger = \text{Table}\left[\begin{cases}\sqrt{i+1} & j == i+1 \\ 0 & \text{True}\end{cases},\ \{j, 0, 4\},\ \{i, 0, 4\}\right]\right)\ //\ \text{MatrixForm}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

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(x = (a + a†)/√2) // MatrixForm
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$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

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(V = α x.x.x.x // Simplify) // MatrixForm
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$$\begin{pmatrix} \frac{3\alpha}{4} & 0 & \frac{3\alpha}{\sqrt{2}} & 0 & \sqrt{\frac{3}{2}}\,\alpha \\ 0 & \frac{15\alpha}{4} & 0 & 5\sqrt{\frac{3}{2}}\,\alpha & 0 \\ \frac{3\alpha}{\sqrt{2}} & 0 & \frac{39\alpha}{4} & 0 & \frac{9\sqrt{3}\,\alpha}{2} \\ 0 & 5\sqrt{\frac{3}{2}}\,\alpha & 0 & \frac{55\alpha}{4} & 0 \\ \sqrt{\frac{3}{2}}\,\alpha & 0 & \frac{9\sqrt{3}\,\alpha}{2} & 0 & 7\alpha \end{pmatrix}$$

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(H = V + DiagonalMatrix[Range[0, 4]]) // MatrixForm
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$$\begin{pmatrix} \frac{3\alpha}{4} & 0 & \frac{3\alpha}{\sqrt{2}} & 0 & \sqrt{\frac{3}{2}}\,\alpha \\ 0 & 1+\frac{15\alpha}{4} & 0 & 5\sqrt{\frac{3}{2}}\,\alpha & 0 \\ \frac{3\alpha}{\sqrt{2}} & 0 & 2+\frac{39\alpha}{4} & 0 & \frac{9\sqrt{3}\,\alpha}{2} \\ 0 & 5\sqrt{\frac{3}{2}}\,\alpha & 0 & 3+\frac{55\alpha}{4} & 0 \\ \sqrt{\frac{3}{2}}\,\alpha & 0 & \frac{9\sqrt{3}\,\alpha}{2} & 0 & 4+7\alpha \end{pmatrix}$$

5c reorder

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in = {2, 4, 1, 3, 5};
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(H = H[[in, in]]) // MatrixForm

$$\begin{pmatrix} 1+\frac{15\alpha}{4} & 5\sqrt{\frac{3}{2}}\,\alpha & 0 & 0 & 0 \\ 5\sqrt{\frac{3}{2}}\,\alpha & 3+\frac{55\alpha}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{3\alpha}{4} & \frac{3\alpha}{\sqrt{2}} & \sqrt{\frac{3}{2}}\,\alpha \\ 0 & 0 & \frac{3\alpha}{\sqrt{2}} & 2+\frac{39\alpha}{4} & \frac{9\sqrt{3}\,\alpha}{2} \\ 0 & 0 & \sqrt{\frac{3}{2}}\,\alpha & \frac{9\sqrt{3}\,\alpha}{2} & 4+7\alpha \end{pmatrix}$$

in = {3, 4, 5};

eval = Eigenvalues[H[[in, in]]]

$\{\text{Root}[-96\alpha - 300\alpha^2 + (128 + 920\alpha + 225\alpha^2)\,\#1 + (-96 - 280\alpha)\,\#1^2 + 16\,\#1^3\ \&,\ 1],$
$\text{Root}[-96\alpha - 300\alpha^2 + (128 + 920\alpha + 225\alpha^2)\,\#1 + (-96 - 280\alpha)\,\#1^2 + 16\,\#1^3\ \&,\ 2],$
$\text{Root}[-96\alpha - 300\alpha^2 + (128 + 920\alpha + 225\alpha^2)\,\#1 + (-96 - 280\alpha)\,\#1^2 + 16\,\#1^3\ \&,\ 3]\}$

Plot$\left[\left\{\text{eval}[[1]],\ \frac{3\alpha}{4} - \frac{21\alpha^2}{8}\right\},\ \{\alpha, 0, .2\}\right]$