Superconducting Diode Effect

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1 Homogeneous Ginzburg-Landau Theory & Second-Order Phase Transitions

We begin by considering a spatially homogeneous superconductor. Near the critical transition temperature T_c, the superconducting state is macroscopically described by a complex order parameter,

$$\psi = |\psi|e^{i\phi} \tag{1}$$

where $n_s = 2|\psi|^2$ represents the local density of superconducting Cooper pairs.

Following Landau's general theory of second-order phase transitions, we expand the Helmholtz free energy density of the superconducting state (F_s) relative to the normal state (F_n) in even powers of the scalar magnitude $|\psi|$:

$$F_s - F_n = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \tag{2}$$

where $\beta > 0$ is a positive constant ensuring the thermodynamic stability of the system at high field values. The quadratic coefficient α drives the phase transition and is phenomenologically parameterized as a linear function of temperature near T_c:

$$\alpha(T) = \alpha_0(T - T_c), \quad (\alpha_0 > 0) \tag{3}$$

1.1 Thermodynamic Minimization

To find the equilibrium state of the system, we minimize the free energy density with respect to the order parameter magnitude, $\frac{\partial F}{\partial|\psi|} = 0$:

$$2\alpha|\psi| + 2\beta|\psi|^3 = 2|\psi|(\alpha + \beta|\psi|^2) = 0 \tag{4}$$

This optimization yields two physically distinct regimes based upon the sign of α:

1. **Normal State ($T > T_c \implies \alpha > 0$):** The only physically real root is $|\psi|_0 = 0$. The system minimizes its energy by remaining in the normal metallic phase.

2. **Superconducting State ($T < T_c \implies \alpha < 0$):** The minimum shifts continuously away from zero to a stable, non-zero bulk expectation value:

$$|\psi|_0^2 = -\frac{\alpha}{\beta} = \frac{\alpha_0(T_c - T)}{\beta} \tag{5}$$

Substituting the stable superconducting bulk value back into the Landau expansion yields the equilibrium condensation energy density:

$$F_s - F_n = -\frac{\alpha^2}{2\beta} \tag{6}$$

Because the order parameter continuously grows from zero starting at $T = T_c$, this represents a textbook **second-order phase transition** accompanied by the spontaneous breaking of a continuous $U(1)$ gauge symmetry.

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2 Inhomogeneous Generalization & Variational Derivation of the GL Equations

To incorporate spatial configurations and non-local effects, we introduce spatial gradients. In the presence of an external magnetic field characterized by the vector potential $\boldsymbol{A}$ (where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$), the gauge-invariant Ginzburg-Landau total free energy functional $G[\psi, \psi^*, \boldsymbol{A}]$ is written as:

$$G = \int d^3r \left[\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*} \left| \left(-i\hbar\nabla - \frac{2e}{c}\boldsymbol{A} \right) \psi \right|^2 + \frac{\boldsymbol{B}^2}{8\pi} \right] \tag{7}$$

where $m^* = 2m_e$ is the effective mass of a Cooper pair and $q^* = 2e$ is its charge.

2.1 Variational Derivation of the First GL Equation

We minimize the total functional by taking an independent variation with respect to the complex conjugate field $\psi^* \to \psi^* + \delta\psi^*$. The variation yields:

$$\delta G_{\psi^*} = \int d^3r \left[\alpha\psi\delta\psi^* + \beta|\psi|^2\psi\delta\psi^* + \frac{1}{2m^*} \left(-i\hbar\nabla - \frac{2e}{c}\boldsymbol{A} \right) \psi \cdot \left(i\hbar\nabla - \frac{2e}{c}\boldsymbol{A} \right) \delta\psi^* \right] \tag{8}$$

To isolate $\delta\psi^*$, we apply integration by parts to the gradient term using the vector identity $\nabla \cdot (\boldsymbol{v} \cdot u) = (\nabla \cdot \boldsymbol{v})u + \boldsymbol{v} \cdot \nabla u$:

$$\int d^3r\, \boldsymbol{v} \cdot \nabla(\delta\psi^*) = \oint d\boldsymbol{S} \cdot \boldsymbol{v}(\delta\psi^*) - \int d^3r(\nabla \cdot \boldsymbol{v})\delta\psi^* \tag{9}$$

Imposing $\delta G = 0$ for arbitrary internal variations $\delta\psi^*$ leads directly to the **First Ginzburg-Landau Equation**:

$$\frac{1}{2m^*} \left(-i\hbar\nabla - \frac{2e}{c}\boldsymbol{A} \right)^2 \psi + \alpha\psi + \beta|\psi|^2\psi = 0 \tag{10}$$

subject to the natural, zero-current boundary condition at the boundaries of the specimen:

$$\boldsymbol{n} \cdot \left(-i\hbar\nabla - \frac{2e}{c}\boldsymbol{A} \right) \psi = 0 \tag{11}$$

2.2 Variational Derivation of the Supercurrent Density (Second GL Equation)

Next, we perform a variation of the functional with respect to the vector potential components $\boldsymbol{A} \to \boldsymbol{A} + \delta\boldsymbol{A}$. Varying the magnetic energy component ($\boldsymbol{B}^2/8\pi$) and utilizing $\delta\boldsymbol{B} = \nabla \times \delta\boldsymbol{A}$ along with integration by parts maps the field variation to $\frac{1}{4\pi} \int d^3r(\nabla \times \boldsymbol{B}) \cdot \delta\boldsymbol{A}$.

Varying the kinetic gradient block with respect to $\boldsymbol{A}$ yields:

$$\delta G_{\boldsymbol{A}} = \int d^3r \left[-\frac{2e}{2m^*c} \left(\psi^* \left(-i\hbar\nabla - \frac{2e}{c}\boldsymbol{A} \right) \psi + \psi \left(i\hbar\nabla - \frac{2e}{c}\boldsymbol{A} \right) \psi^* \right) \right] \cdot \delta\boldsymbol{A} \tag{12}$$

Equating the functional derivative to the Maxwellian definition ($\delta G_{\text{field}} = \int d^3r \frac{1}{4\pi c}\boldsymbol{j} \cdot \delta\boldsymbol{A}$ where $\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}$) isolates the **Supercurrent Density Vector ($\boldsymbol{j}$)**:

$$\boldsymbol{j} = \frac{2e\hbar}{2m^*i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{4e^2}{m^*c}|\psi|^2\boldsymbol{A} \tag{13}$$

2.3 The Superflow Velocity

Substituting the explicit polar configuration $\psi(\boldsymbol{r}) = |\psi|e^{i\phi(\boldsymbol{r})}$ directly into the current density equation gives:

$$\boldsymbol{j} = \frac{2e\hbar}{2m^*i}\left(2i|\psi|^2\nabla\phi\right) - \frac{4e^2}{m^*c}|\psi|^2\boldsymbol{A} = \frac{2e}{m^*}|\psi|^2\left(\hbar\nabla\phi - \frac{2e}{c}\boldsymbol{A}\right) \tag{14}$$

This allows us to naturally define the gauge-invariant **superflow velocity ($\boldsymbol{v}_s$)**:

$$\boldsymbol{v}_s = \frac{1}{m^*}\left(\hbar\nabla\phi - \frac{2e}{c}\boldsymbol{A}\right) \tag{15}$$

which simplifies the current density equation to its clean hydrodynamic form:

$$\boldsymbol{j} = 2e|\psi|^2\boldsymbol{v}_s \tag{16}$$

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3 Thermodynamic Depairing Critical Current Density (j_c)

Let us now solve for the absolute maximum sustainable current supported by a bulk superconductor. Consider a uniform, one-dimensional current flowing at a constant velocity $\boldsymbol{v}_s$. The spatial modulation is held purely in the phase factor: $\psi = |\psi|e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$, so that $\hbar\nabla\phi = \hbar\boldsymbol{k}$.

Assuming a thin wire or film where magnetic screening is negligible ($\boldsymbol{A} = 0$), the kinetic energy density contribution expands to:

$$F_{\text{kin}} = \frac{1}{2m^*}|i\hbar\boldsymbol{k}\psi|^2 = \frac{\hbar^2k^2}{2m^*}|\psi|^2 = \frac{1}{2}m^*v_s^2|\psi|^2 \tag{17}$$

The total modified free energy density expression reads:

$$F(|\psi|, v_s) = \left(\alpha + \frac{1}{2}m^*v_s^2\right)|\psi|^2 + \frac{\beta}{2}|\psi|^4 \tag{18}$$

3.1 Minimization at Constrained Velocity

Minimizing this expression with respect to the density $|\psi|^2$ at a fixed superflow velocity v_s yields:

$$|\psi(v_s)|^2 = -\frac{1}{\beta}\left(\alpha + \frac{1}{2}m^*v_s^2\right) = |\psi|_0^2\left(1 - \frac{m^*v_s^2}{2|\alpha|}\right) \tag{19}$$

where $|\psi|_0^2 = -\alpha/\beta$ is the resting zero-velocity bulk density. This shows that the internal order parameter density is continuously depleted by the kinetic energy of the flow.

3.2 Maximizing the Current Density

We substitute this velocity-dependent density profile back into the supercurrent equation:

$$j(v_s) = 2e|\psi(v_s)|^2v_s = 2e|\psi|_0^2\left(v_s - \frac{m^*v_s^3}{2|\alpha|}\right) \tag{20}$$

To determine the maximum current density the system can sustain before breaking superconductivity, we locate the extreme value via $\frac{dj}{dv_s} = 0$:

$$\frac{dj}{dv_s} = 2e|\psi|_0^2\left(1 - \frac{3m^*v_s^2}{2|\alpha|}\right) = 0 \implies v_c = \sqrt{\frac{2|\alpha|}{3m^*}} \tag{21}$$

At this critical velocity threshold (v_c), the Cooper pair density has dropped to exactly $\frac{2}{3}|\psi|_0^2$. Substituting v_c back into $j(v_s)$ gives the standard **thermodynamic depairing critical current density (j_c)**:

$$j_c = \frac{4}{3\sqrt{3}} e|\psi|_0^2 \sqrt{\frac{2|\alpha|}{3m^*}} \tag{22}$$

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4 Lifshitz Invariants, Helical States, and the Magnetoelectric Effect

In materials or interfaces where **spatial inversion symmetry is broken** (non-centrosymmetric structures), the free energy expansion can include terms containing an *odd* number of spatial derivatives. These symmetry-allowed gradient terms are called **Lifshitz invariants**.

In the presence of an external in-plane magnetic field $\boldsymbol{B}$, a linear derivative term couples the field to the supercurrent. For a one-dimensional channel directed along $\hat{x}$ under a perpendicular in-plane field B_y, the free energy density is augmented by:

$$F_{\text{Lifshitz}} = \gamma B_y \left[\psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \psi + \text{c.c.} \right] = 2\gamma\hbar B_y |\psi|^2 \frac{\partial \phi}{\partial x} \tag{23}$$

4.1 Stabilization of the Helical State

Compiling the total free energy density including this linear invariant (setting $\boldsymbol{A} = 0$ and assuming a uniform magnitude $|\psi|$) yields:

$$F = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{2m^*}|\psi|^2 \left(\frac{\partial \phi}{\partial x} \right)^2 + 2\gamma\hbar B_y |\psi|^2 \frac{\partial \phi}{\partial x} \tag{24}$$

Minimizing this expression with respect to the static phase gradient $q_x = \frac{\partial \phi}{\partial x}$ gives:

$$\frac{\partial F}{\partial q_x} = \frac{\hbar^2}{2m^*} 2q_x |\psi|^2 + 2\gamma\hbar B_y |\psi|^2 = 0 \implies q_0 = -\frac{2m^* \gamma B_y}{\hbar} \tag{25}$$

Because the absolute energy minimum occurs at a non-zero phase gradient ($q_0 \neq 0$), the ground state of the superconductor is no longer uniform. Instead, the phase winds spontaneously in space, forming a **helical state**:

$$\psi(x) = |\psi| e^{iq_0 x} \tag{26}$$

4.2 Magnetoelectric Effect

By performing a variational minimization of this complete functional with respect to the magnetic field components, we reveal a modified Ginzburg-Landau constitutive relation. The presence of the Lifshitz invariant dictates that driving a supercurrent ($\boldsymbol{v}_s$) through the system directly induces a finite, macroscopic magnetization ($\boldsymbol{M}$):

$$\boldsymbol{M}_{\text{induced}} \propto \gamma |\psi|^2 \boldsymbol{v}_s \tag{27}$$

This cross-coupling between an electrical superflow and magnetic polarization is known as the **superconducting magnetoelectric effect**.

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5 Phenomenology of the Superconducting Diode Effect (SDE)

To establish a **Superconducting Diode Effect (SDE)**, a system must simultaneously break both **spatial inversion symmetry ($\mathcal{P}$)** and **time-reversal symmetry ($\mathcal{T}$)**. While a linear Lifshitz invariant ($\alpha_1 q$) breaks $\mathcal{P}$ and couples to a $\mathcal{T}$-breaking field, it merely shifts the global energy minimum to q_0. The energy landscape remains perfectly symmetric about this new center: $E(q_0+\Delta q) = E(q_0-\Delta q)$, keeping the critical current completely reciprocal ($j_{c+} = |j_{c-}|$).

Following the general framework outlined in *arXiv:2404.17072 (Section II)*, we generalize the Ginzburg-Landau expansion by allowing both the quadratic (α) and quartic (β) coefficients to depend explicitly on the Cooper pair momentum vector $\boldsymbol{q}$:

$$F = \alpha(\boldsymbol{q})|\psi|^2 + \frac{\beta(\boldsymbol{q})}{2}|\psi|^4 \tag{28}$$

We expand $\alpha(\boldsymbol{q})$ and $\beta(\boldsymbol{q})$ in powers of $\boldsymbol{q}$ up to cubic order, where the odd powers are linear in the $\mathcal{T}$-breaking magnetic field:

$$\alpha(\boldsymbol{q}) = \alpha_0 + \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 \tag{29}$$

$$\beta(\boldsymbol{q}) = \beta_0 + \beta_1 q + \beta_2 q^2 + \beta_3 q^3 \tag{30}$$

We assume the isotropic, symmetric terms dominate ($\alpha_2 > 0, \beta_0 > 0$) and treat the nonreciprocal odd terms ($\alpha_1, \alpha_3, \beta_1, \beta_3 \propto B$) as small perturbations.

5.1 Derivation of the Momentum-Dependent Supercurrent

Minimizing the free energy with respect to $|\psi|^2$ gives the equilibrium density profile at a given momentum q:

$$|\psi(q)|^2 = -\frac{\alpha(q)}{\beta(q)} \tag{31}$$

The current density is obtained by taking the derivative of the free energy density with respect to the momentum vector q:

$$j(q) = \frac{2e}{\hbar}\frac{\partial F}{\partial q} = \frac{2e}{\hbar}\left[\alpha'(q)|\psi|^2 + \frac{1}{2}\beta'(q)|\psi|^4\right] \tag{32}$$

Substituting the equilibrium density profile $|\psi(q)|^2 = -\alpha(q)/\beta(q)$ into the current density equation yields:

$$j(q) = -\frac{2e}{\hbar}\frac{\alpha(q)}{\beta(q)}\left[\alpha'(q) - \frac{\alpha(q)\beta'(q)}{2\beta(q)}\right] \tag{33}$$

5.2 Evaluation of the Critical Currents j_{c+} and j_{c-}

The forward (j_{c+}) and reverse (j_{c-}) critical currents correspond to the maximum and minimum values of $j(q)$, found by setting $\frac{dj}{dq} = 0$.

The unperturbed symmetric current is $j_0(q) = -\frac{2e}{\hbar}\frac{\alpha_0+\alpha_2 q^2}{\beta_0}(2\alpha_2 q)$, which peaks at the unperturbed critical momentum $q_c^{(0)} = \sqrt{\frac{-\alpha_0}{3\alpha_2}}$. Expanding $j(q)$ to first order in the perturbation coefficients around $\pm q_c^{(0)}$ gives the asymmetric critical current thresholds:

$$j_{c+} = j_c^{(0)} + \delta j, \quad |j_{c-}| = j_c^{(0)} - \delta j \tag{34}$$

where the absolute current asymmetry parameter $\delta j = \frac{j_{c+}-|j_{c-}|}{2}$ evaluates to:

$$\delta j = \frac{4e}{3\sqrt{3}\hbar}\frac{(-\alpha_0)^{3/2}}{\beta_0\sqrt{\alpha_2}}\left[\frac{\alpha_3}{\alpha_2} - \frac{\alpha_1\beta_2}{\alpha_2\beta_0} - \frac{\alpha_0}{2\alpha_2}\left(\frac{\beta_3}{\beta_0} - \frac{\beta_1\beta_2}{\beta_0^2}\right)\right] \tag{35}$$

5.3 Diode Efficiency

The efficiency of the superconducting diode is given by the dimensionless ratio η:

$$\eta = \frac{j_{c+} - |j_{c-}|}{j_{c+} + |j_{c-}|} = \frac{\delta j}{j_c^{(0)}} \propto \left[\frac{\alpha_3}{\alpha_2} - \frac{\alpha_1 \beta_2}{\alpha_2 \beta_0} - \frac{\alpha_0}{2\alpha_2} \left(\frac{\beta_3}{\beta_0} - \frac{\beta_1 \beta_2}{\beta_0^2} \right) \right] \tag{36}$$

5.4 Crucial Insight: Why a Linear Lifshitz Term (α_1) Alone is Insufficient

This result highlights an essential physical constraint of the superconducting diode effect: **A purely linear momentum term ($\alpha_1 \neq 0$) cannot generate a diode effect on its own if the quartic parameter remains unperturbed ($\beta_1 = \beta_3 = 0$) and higher-order gradients are neglected ($\alpha_3 = 0$).**

Under those conditions, the asymmetry expression reduces to:

$$\delta j \propto -\frac{\alpha_1 \beta_2}{\alpha_2 \beta_0} \tag{37}$$

If the quartic parameter is entirely independent of momentum ($\beta_2 = 0$), then $\delta j = 0$ and $\eta = 0$.

Physically, this occurs because a purely linear term $\alpha_1 q$ can be absorbed into the quadratic sector by completing the square:

$$\alpha_2 q^2 + \alpha_1 q = \alpha_2 (q - q_0)^2 - \frac{\alpha_1^2}{4\alpha_2} \tag{38}$$

where $q_0 = -\alpha_1/2\alpha_2$. If the rest of the functional (β_0) does not depend on the direction of q, this transformation simply shifts the origin of momentum space to the helical wavevector q_0. The resulting current profile remains perfectly symmetric about q_0, meaning the maximum forward current matches the reverse current:

$$j_{\max} = j(q_0 + q_c^{(0)}) = -j(q_0 - q_c^{(0)}) \implies j_{c+} = |j_{c-}| \tag{39}$$

Therefore, a non-zero superconducting diode efficiency ($\eta \neq 0$) fundamentally requires **higher-order structural deformations of the momentum landscape**, which manifest as either:

- Cubic momentum corrections to the kinetic energy ($\alpha_3 \neq 0$, coming from higher-order gradient Lifshitz invariants).
- Momentum dependence in the quartic parameter ($\beta_1, \beta_3 \neq 0$), reflecting nonreciprocal pairing interactions as the superconducting gap opens.