Lecture 15

Goals:

• Chapter 11
  - Employ the dot product
  - Employ conservative and non-conservative forces
  - Use the concept of power (i.e., energy per time)
• Chapter 12
  - Extend the particle model to rigid-bodies
  - Understand the equilibrium of an extended object.
  - Understand rigid object rotation about a fixed axis.
  - Employ “conservation of angular momentum” concept

Assignment:
• HW7 due March 10th
• For Thursday: Read Chapter 12, Sections 7-11
do not concern yourself with the integration process in regards to “center of mass” or “moment of inertia”

Scalar Product (or Dot Product)

\[ \mathbf{A} \cdot \mathbf{B} \equiv | \mathbf{A} | | \mathbf{B} | \cos \theta \]

• Useful for finding parallel components
  \[ \mathbf{A} \cdot \mathbf{i} = A_x \]
  \[ \mathbf{i} \cdot \mathbf{j} = 1 \]
  \[ \mathbf{i} \cdot \mathbf{j} = 0 \]

• Calculation can be made in terms of components.
  \[ \mathbf{A} \cdot \mathbf{B} = (A_x)(B_x) + (A_y)(B_y) + (A_z)(B_z) \]

Calculation also in terms of magnitudes and relative angles.
  \[ \mathbf{A} \cdot \mathbf{B} \equiv | \mathbf{A} | | \mathbf{B} | \cos \theta \]

*You choose the way that works best for you!*

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Scalar Product (or Dot Product)

Compare:
\[ \mathbf{A} \cdot \mathbf{B} = (A_x)(B_x) + (A_y)(B_y) + (A_z)(B_z) \]

Redefine \( \mathbf{A} \rightarrow \mathbf{F} \) (force), \( \mathbf{B} \rightarrow \Delta \mathbf{r} \) (displacement)

Notice:
\[ \mathbf{F} \cdot \Delta \mathbf{r} = (F_x)(\Delta x) + (F_y)(\Delta z) \]

So here
\[ \mathbf{F} \cdot \Delta \mathbf{r} = W \]

More generally a Force acting over a Distance does Work

Work in terms of the dot product

**Ingredients:** Force (\( \mathbf{F} \)), displacement (\( \Delta \mathbf{r} \))

Work, \( W \), of a constant force \( \mathbf{F} \) acts through a displacement \( \Delta \mathbf{r} \):

\[ W = |\mathbf{F}| \cos \theta |\Delta \mathbf{r}| = \mathbf{F} \cdot \Delta \mathbf{r} \]

Looks just like a Dot Product!

If the path is curved \( dW = \mathbf{F} \cdot d\mathbf{r} \) at each point and

\[ W = \int_{\vec{r}_i}^{\vec{r}_f} \mathbf{F} \cdot d\mathbf{r} \]
Remember that a real trajectory implies forces acting on an object

\[
\mathbf{a} = a_\parallel + a_\perp
\]

\[
\mathbf{F} = F_{\text{tang}} + F_{\text{radial}}
\]

Two possible options:
- Change in the magnitude of \(\mathbf{v}\)
- Change in the direction of \(\mathbf{a}\)

Only tangential forces yield work!
- The distance over which \(F_{\text{Tang}}\) is applied: Work

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**Energy and Work**

Work, \(W\), is the process of energy transfer in which a force component parallel to the path acts over a distance; individually it effects a change in energy of the “system”.

1. K or Kinetic Energy
2. U or Potential Energy (Conservative)

and if there are losses (e.g., friction, non-conservative)

3. \(E_{\text{Th}}\) Thermal Energy

Positive \(W\) if energy transferred to a system
A child slides down a playground slide at constant speed. The energy transformation is

A. $U \rightarrow K$
B. $U \rightarrow E_{Th}$
C. $K \rightarrow U$
D. $K \rightarrow E_{Th}$
E. There is no transformation because energy is conserved.

**Exercise**

Work in the presence of friction and non-contact forces

- A box is pulled up a rough ($\mu > 0$) incline by a rope-pulley-weight arrangement as shown below.
  - How many forces (including non-contact ones) are doing work on the box?
  - Of these which are positive and which are negative?
  - State the system (here, just the box)
  - Use a Free Body Diagram
  - Compare force and path

A. 2
B. 3
C. 4
D. 5
Work and Varying Forces (1D)

- Consider a varying force $F(x)$

\[ \text{Area} = F \cdot \Delta x \]

$F$ is increasing

Here \( W = F \cdot \Delta r \)

becomes \( dW = F \cdot dx \)

\[ W = \int_{x_i}^{x_f} F(x) \, dx \]

\[ \text{Work has units of energy and is a scalar!} \]

Example: Hooke’s Law Spring \((x_i \text{ equilibrium})\)

- How much will the spring compress (i.e. \( \Delta x = x_f - x_i \)) to bring the box to a stop (i.e., \( v = 0 \)) if the object is moving initially at a constant velocity \((v_i)\) on frictionless surface as shown below with \( x_i = x_{eq} \), the equilibrium position of the spring?

\[ W_{\text{box}} = \int_{x_i}^{x_f} F(x) \, dx \]

\[ W_{\text{box}} = \int_{x_i}^{x_f} -k(x - x_{eq}) \, dx \]

\[ W_{\text{box}} = -\frac{1}{2} k(x_f - x_i)^2 |_{x_i}^{x_f} \]

\[ W_{\text{box}} = -\frac{1}{2} k(x_f - x_i)^2 + \frac{1}{2} k0^2 = \Delta K \]

\[ \frac{1}{2} k \Delta x^2 = \frac{1}{2} m0^2 - \frac{1}{2} m v_i^2 \]
Work signs

Notice that the spring force is opposite the displacement

For the mass $m$, work is negative

For the spring, work is positive

They are opposite, and equal (spring is conservative)

Conservative Forces & Potential Energy

- For any conservative force $F$ we can define a potential energy function $U$ in the following way:

$$W = \int F \cdot dr \equiv -\Delta U$$

The work done by a conservative force is equal and opposite to the change in the potential energy function.
Conservative Forces and Potential Energy

- So we can also describe work and changes in potential energy (for conservative forces)
  \[ \Delta U = -W \]

- Recalling (if 1D)
  \[ W = F_x \Delta x \]

- Combining these two,
  \[ \Delta U = -F_x \Delta x \]

- Letting small quantities go to infinitesimals,
  \[ dU = -F_x dx \]

- Or,
  \[ F_x = -\frac{dU}{dx} \]

Exercise

Work Done by Gravity

- An frictionless track is at an angle of 30° with respect to the horizontal. A cart (mass 1 kg) is released from rest. It slides 1 meter downwards along the track bounces and then slides upwards to its original position.

- How much total work is done by gravity on the cart when it reaches its original position? (g = 10 m/s^2)

   \[ h = 1 \text{ m} \sin 30° = 0.5 \text{ m} \]

   (A) 5 J    (B) 10 J    (C) 20 J    (D) 0 J
Home Exercise: Work & Friction

- Two blocks having mass \( m_1 \) and \( m_2 \) where \( m_1 > m_2 \). They are sliding on a frictionless floor and have the same kinetic energy when they encounter a long rough stretch (i.e. \( \mu > 0 \)) which slows them down to a stop.

- Which one will go farther before stopping?
- **Hint:** How much work does friction do on each block?

  (A) \( m_1 \)  
  (B) \( m_2 \)  
  (C) They will go the same distance

Exercise: Work & Friction

- \( W = F d = -\mu N d = -\mu mg d = \Delta K = 0 - \frac{1}{2} mv^2 \)

- \( -\mu m_1g d_1 = -\mu m_2g d_2 \rightarrow d_1 / d_2 = m_2 / m_1 \)

  (A) \( m_1 \)  
  (B) \( m_2 \)  
  (C) They will go the same distance
Home Exercise
Work/Energy for Non-Conservative Forces

● The air track is once again at an angle of 30° with respect to horizontal. The cart (with mass 1 kg) is released 1 meter from the bottom and hits the bumper at a speed, $v_1$. This time the vacuum/air generator breaks half-way through and the air stops. The cart only bounces up half as high as where it started.

● How much work did friction do on the cart? ($g=10 \text{ m/s}^2$)

\[
W = F \Delta x \text{ is not easy to do…}
\]

Work done is equal to the change in the energy of the system (U and/or K). $E_{\text{final}} - E_{\text{initial}}$ and is < 0. ($E = U+K$)

Use $W = U_{\text{final}} - U_{\text{initial}} = mg (h_f - h_i) = -mg \sin 30° \times 0.5 \text{ m}$

$W = -2.5 \text{ N m} = -2.5 \text{ J}$ or (D)

(A) 2.5 J  (B) 5 J  (C) 10 J  (D) −2.5 J  (E) −5 J  (F) −10 J
A Non-Conservative Force

Since \( \text{path}_2 \) distance > \( \text{path}_1 \) distance the puck will be traveling slower at the end of \( \text{path}_2 \).

Work done by a non-conservative force irreversibly removes energy out of the "system".

Here \( W_{\text{NC}} = E_{\text{final}} - E_{\text{initial}} < 0 \) \( \rightarrow \) and reflects \( E_{\text{thermal}} \)

Work & Power:

- Two cars go up a hill, a Corvette and a ordinary Chevy Malibu. Both cars have the same mass.
- Assuming identical friction, both engines do the same amount of work to get up the hill.
- Are the cars essentially the same?
- NO. The Corvette can get up the hill quicker
- It has a more powerful engine.
Work & Power:

- Power is the rate at which work is done.

<table>
<thead>
<tr>
<th>Average Power:</th>
<th>Instantaneous Power:</th>
<th>Units (SI) are Watts (W):</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P = \frac{W}{\Delta t}$</td>
<td>$P = \frac{dW}{dt}$</td>
<td>$1 \text{ W} = 1 \text{ J} / \text{ s}$</td>
</tr>
</tbody>
</table>

Example:

- A person, mass 80.0 kg, runs up 2 floors (8.0 m). If they climb it in 5.0 sec, what is the average power used?
  - $P_{\text{avg}} = \frac{F \cdot h}{\Delta t} = \frac{mgh}{\Delta t} = \frac{80.0 \times 9.80 \times 8.0}{5.0} \text{ W}$
  - $P = 1250 \text{ W}$

Work & Power:

- Power is also, $\overline{P} = \frac{W}{\Delta t} = \frac{\int F \, dx}{\Delta t} \rightarrow P = F_x \, v_x$

- If force constant, $W = F \, \Delta x = F \left( v_0 \, \Delta t + \frac{1}{2} a \, \Delta t^2 \right)$
  and $P = \frac{W}{\Delta t} = F \left( v_0 + a \Delta t \right)$
**Exercise**

**Work & Power**

- Starting from rest, a car drives up a hill at constant acceleration and then quickly stops at the top.

(Hint: What does constant acceleration imply?)

- The instantaneous power delivered by the engine during this drive looks like which of the following,

  A. Top
  B. Middle
  C. Bottom

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**Chap. 12: Rotational Dynamics**

- Up until now rotation has been only in terms of circular motion with $a_c = \frac{v^2}{R}$ and $|a_T| = \frac{d|v|}{dt}$
- Rotation is common in the world around us.
- Many ideas developed for translational motion are transferable.
Rotational Variables

- Rotation about a fixed axis:
  - Consider a disk rotating about an axis through its center:

- Recall:
  \[
  \omega = \frac{d\theta}{dt} = \frac{2\pi}{T} \quad \text{(rad/s)} = \frac{v_{\text{Tangential}}}{R}
  \]
  (Analogous to the linear case \( v = \frac{dx}{dt} \))

Rotational Variables...

At a point a distance \( R \) away from the axis of rotation, the tangential motion:

- \( x \, (\text{arc}) = \theta \, R \)
- \( v_T \, (\text{tangential}) = \omega \, R \)
- \( a_T = \alpha \, R \)

\( \alpha = \text{constant} \) (angular acceleration in rad/s\(^2\))
\( \omega = \omega_0 + \alpha \Delta t \) (angular velocity in rad/s)
\( \theta = \theta_0 + \omega_0 \Delta t + \frac{1}{2} \alpha \Delta t^2 \) (angular position in rad)
Overview (with comparison to 1-D kinematics)

<table>
<thead>
<tr>
<th>Angular</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = \text{constant} )</td>
<td>( a = \text{constant} )</td>
</tr>
<tr>
<td>( \omega = \omega_0 + \alpha \Delta t )</td>
<td>( v = v_0 + a \Delta t )</td>
</tr>
<tr>
<td>( \theta = \theta_0 + \omega_0 \Delta t + \frac{1}{2} \alpha \Delta t^2 )</td>
<td>( x = x_0 + v_0 \Delta t + \frac{1}{2} a \Delta t^2 )</td>
</tr>
</tbody>
</table>

And for a point at a distance \( R \) from the rotation axis:

\[
x = R \theta \\
v = \omega R \\
a_T = \alpha R
\]

Here \( a_T \) refers to tangential acceleration

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